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SCATTERING THEORY FOR DIFFRACTION GRATINGS.(U)

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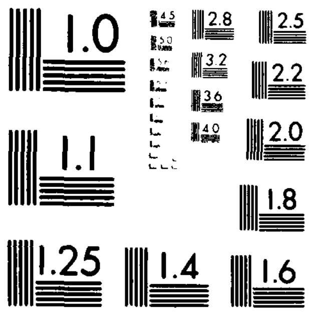

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(6) SCATTERING THEORY FOR DIFFRACTION GRATINGS.

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Abstract.

Scattering theory is developed for plane diffraction gratings. The author's theory of Rayleigh-Bloch wave expansions is used to construct wave operators and a scattering operator for such gratings. For gratings that admit no surface waves, transient wave fields near the gratings are shown to behave for large times like free waves and corresponding asymptotic wave functions are calculated. These results are applied to analyze the echoes from gratings of signals due to localized sources. Finally, the echoes of sources remote from the grating are estimated and shown to be completely characterized by the S-matrix and the signal waveform.

Introduction.

A theory of the scattering of acoustic and electromagnetic waves by plane diffraction gratings was initiated in [5, 6]. That work developed the theory of Rayleigh-Bloch (or R-B) waves for diffraction gratings. Physically, the R-B waves describe the response of a diffraction grating to an incident monochromatic plane wave. The principal result of [5, 6] is an R-B wave expansion of arbitrary wave fields. The purpose of the present report is to apply the results of [5, 6] to an analysis of the scattering of transient wave fields by diffraction gratings. To simplify the analysis it is assumed throughout this report that the grating propagator A(G) admits no R-B surface waves. In the general case most of the results of this report hold for states orthogonal to the subspace spanned by the surface waves (see [5, (6.36)]). The scattering of R-B surface waves is not analyzed in this report.

The report is organized as follows. §1 presents a review of the initial-boundary value problem for transient wave fields near a diffraction grating and its solution by means of the Hilbert space theory of the grating propagator developed in [1, 3]. §2 presents a construction of the wave operators for the pair consisting of the reduced propagator A_p of [5] and the reduced propagator $A_{0,p}$ for the degenerate grating. The results of §2 are used in §3 to construct the wave operators for the pair consisting of the grating propagator A and the propagator A_0 of the degenerate grating. The results of §3 imply that transient wave fields near gratings have asymptotic wave functions in the sense of the author's monograph on scattering by bounded obstacles [3]. These functions, and the corresponding asymptotic energy distributions are derived in §4. In

§5 the S-matrix for the pair A and A_0 is calculated. §6 contains an analysis of the structure of the echoes produced by sources that are far from the grating. The analysis shows that in this case the echo waveform is completely determined by the S-matrix and the signal waveform, just as in the case of scattering by a bounded obstacle [4].

§1. The Initial-Boundary Value Problem for the Scattered Fields.

The notation of [5, 6] will be used in this report. In particular, $X = (x,y) \in R^2$ and G is a domain in R^2. Each transient acoustic or electromagnetic wave field in G can be described by a real-valued potential function $u = u(t,X)$ that is a solution of the initial-boundary value problem

$$(1.1) \qquad D_t^2 u - \Delta u = 0 \text{ for all } t \geq 0 \text{ and } X \in G ,$$

$$(1.2) \qquad D_\nu u \equiv \vec{\nu} \cdot \nabla u = 0 \text{ (resp., } u = 0) \text{ for all } t \geq 0,\ X \in \partial G ,$$

$$(1.3) \qquad u(0,X) = f(X) \text{ and } D_t u(0,X) = g(X) \text{ for all } X \in G .$$

Here $t \in R$ is a time-coordinate, $\nabla u = (D_x u, D_y u)$, $\Delta u = D_x^2 u + D_y^2 u$, ∂G denotes the boundary of G and $\vec{\nu}(X)$ is a unit normal vector to ∂G at X. $u(t,X)$ may be interpreted as a potential for an acoustic field with velocity $\vec{v} = \nabla u$ and excess pressure $p = D_t u$. With this interpretation the boundary condition (1.2) corresponds to an acoustically hard (resp., soft) boundary. Alternatively, if u satisfies the Neumann condition then

$$(1.4) \qquad E_x = D_y u \text{ , } E_y = -D_x u \text{ , } H_z = D_t u$$

describes a TM electromagnetic field in a domain G bounded by a perfect electrical conductor. Similarly, if u satisfies the Dirichlet boundary condition then

$$(1.5) \qquad H_x = -D_y u \text{ , } H_y = D_x u \text{ , } E_z = D_t u$$

describes a TE electromagnetic field in the same kind of domain. In

both the acoustic and electromagnetic interpretations the integral

$$E(u,K,t) = \int_K \{(D_x u)^2 + (D_y u)^2 + (D_t u)^2\}\, dxdy \tag{1.6}$$

is interpreted as the wave energy in the set K at time t. Note that all the physically observable quantities are determined by the derivatives of the potential u. The state of the wave field at the initial time $t = 0$ is defined by the initial conditions (1.3).

The initial-boundary value problem in its classical formulation (1.1) - (1.3) will have a solution only if ∂G and the functions $f(X)$ and $g(X)$ are sufficiently smooth. However, for arbitrary domains G the problem is known to have a unique generalized solution with locally finite energy whenever the initial state f, g has this property. This result was proved in [1]. In cases where the initial state has finite energy,

$$\int_G \{(D_x f)^2 + (D_y f)^2 + g^2\}\, dxdy < \infty , \tag{1.7}$$

a simple approach to the initial-boundary value problem is provided by the grating propagator

$$A = A^N(G) \text{ (resp., } A^D(G)) \tag{1.8}$$

of [5, §1]. For arbitrary domains $G \subset R^2$, A is a selfadjoint realization in the Hilbert space $L_2(G)$ of the operator $-\Delta$. Moreover, $A \geq 0$ and $D(A^{1/2}) = L_2^1(G)$ (resp., $L_2^D(G)$); see [5, §1]. It follows that if

$$f \in L_2^1(G) \text{ (resp., } L_2^D(G)) \text{ and } g \in L_2(G) \tag{1.9}$$

then (1.7) holds and

$$u(t,\cdot) = (\cos t\, A^{1/2})\, f + (A^{-1/2} \sin t\, A^{1/2})\, g \tag{1.10}$$

is the unique solution with finite energy (= solutions wFE) of (1.1) - (1.3); see [3, 5]. In particular,

$$u \in C^1(R, L_2(G)) \cap C(R, D(A^{1/2})) \tag{1.11}$$

and the initial conditions hold in $L_2(G)$. The boundary conditions are incorporated in the definition of D(A). The d'Alembert equation (1.1) holds in a suitable weak form; see [3]. The transient wave fields studied in this report are the solutions wFE defined by (1.9), (1.10).

It was shown in [3, Ch. 3] that solutions wFE in arbitrary domains have a representation

$$u(t,X) = \text{Re}\,\{v(t,X)\}\ ,\ v(t,\cdot) = e^{-itA^{1/2}}\, h \tag{1.12}$$

provided that f and g satisfy (1.9) and $g \in D(A^{-1/2})$. The complex-valued function $h \in D(A^{1/2})$ is related to the initial state f, g by

$$h = f + i\, A^{-1/2}\, g\ . \tag{1.13}$$

This representation is used in §4 below to determine the asymptotic behavior for $t \to \infty$ of the transient wave fields (1.10).

The R-B wave expansions of [5, §6] can be used to construct representations of the solutions wFE (1.10) and (1.12). In the case of (1.12) the representations take the form

$$v(t,X) = \text{l.i.m.} \int_{R_0^2} \psi_\pm(X,P)\, e^{-it\omega(P)}\, \hat{h}_\pm(P)\, dP \tag{1.14}$$

where $P = (p,q)$ and the integral, together with its formal t-derivative, converge in $L_2(G)$ [5, Theorem 6.5 and 6.6].

§2. Construction of the Wave Operators for A_p and $A_{0,p}$.

The notation of [5, §3 and §5] is used in this section. The purpose of the section is to prove the existence and completeness of the wave operators

$$W_{\pm,p} = W_{\pm}(A_{0,p}^{1/2}, A_p^{1/2}, J_\Omega) = \underset{t\to\pm\infty}{\text{s-lim}}\; e^{itA_{0,p}^{1/2}} J_\Omega \, e^{-itA_p^{1/2}} \tag{2.1}$$

where $J_\Omega : L_2(\Omega) \to L_2(B_0)$ is defined by

$$J_\Omega\, h(X) = \begin{cases} h(X) , & X \in \Omega , \\ 0 , & X \in B_0 - \Omega . \end{cases} \tag{2.2}$$

This will be done by means of an explicit construction based on the eigenfunction expansions for A_p and $A_{0,p}$ of [5, §3 and §5]. The principal results are formulated as

Theorem 2.1. Let G be a grating domain [5, §1]. Let $p \in (-1/2,1/2]$ and assume that $\sigma_0(A_p) = \phi$. Then $W_{+,p}$ and $W_{-,p}$ exist and are given by

$$W_{\pm,p} = \Phi_{0,p}^{*}\, \Phi_{\mp,p} . \tag{2.3}$$

In particular, $W_{\pm,p} : L_2(\Omega) \to L_2(B_0)$ are unitary operators and one has

$$\Pi_p(\lambda) = W_{\pm,p}^{*}\, \Pi_{0,p}(\lambda)\, W_{\pm,p} \text{ for all } \lambda \in R . \tag{2.4}$$

Theorem 2.1 is primarily of technical interest in the theory of scattering by diffraction gratings. It will be used in §3 to derive a construction of the wave operators for A and A_0.

Theorem 2.1 will be proved by the method of [3, Ch. 7]. Only the case of $W_{+,p}$ will be discussed, the other case being entirely similar. To begin consider the wave function

$$v(t,\cdot) = e^{-itA_p^{1/2}} h \ , \ h \in L_2(\Omega) \ . \tag{2.5}$$

The eigenfunction expansion theorem for A_p of [5, Theorems 5.6 and 5.7] implies that $v(t,X)$ has the two representations

$$v(t,X) = \underset{M\to\infty}{\text{l.i.m.}} \sum_{|m|\le M} \int_0^M \phi_\pm(X,p+m,q)\, e^{-it\omega(p+m,q)}\, \tilde{h}_\pm(p+m,q)\, dq \ , \tag{2.6}$$

convergent in $L_2(\Omega)$. As in [3, Ch. 7], the incoming representation will be used to calculate the behavior of $v(t,\cdot)$ for $t \to +\infty$. The eigenfunction ϕ_- has the decomposition [5, (5.4) and (5.5)]

$$\phi_-(X,p+m,q) = j(y)\ \phi_0(X,p+m,q) + \phi_-'(X,p+m,q) \tag{2.7}$$

where ϕ_-' is incoming. Combining (2.6) and (2.7) gives

$$v(t,X) = j(y)\ v_0^+(t,X) + v^+(t,X) \tag{2.8}$$

where

$$v_0^+(t,X) = \underset{M\to\infty}{\text{l.i.m.}} \sum_{|m|\le M} \int_0^M \phi_0(X,p+m,q)\, e^{-it\omega(p+m,q)}\, \tilde{h}_-(p+m,q)\, dq \tag{2.9}$$

converges in $L_2(B_0)$ while

$$v^+(t,X) = \underset{M\to\infty}{\text{l.i.m.}} \sum_{|m|\le M} \int_0^M \phi_-'(X,p+m,q)\, e^{-it\omega(p+m,q)}\, \tilde{h}_-(p+m,q)\, dq \tag{2.10}$$

converges in $L_2(\Omega)$. Note that the convergence of (2.6) and (2.9) implies

that of (2.10). Moreover, $v_0^+(t,X)$ is a wave function in $L_2(B_0)$ for the reduced propagator $A_{0,p}$ of the degenerate grating; namely,

$$v_0^+(t,\cdot) = e^{-itA_{0,p}^{1/2}} h_0^+ \tag{2.11}$$

where $h_0^+ = v_0^+(0,\cdot) \in L_2(B_0)$ is given by

$$h_0^+ = \Phi_{0,p}^* \tilde{h}_- = \Phi_{0,p}^* \Phi_{-,p} h . \tag{2.12}$$

Theorem 2.1 will be shown to be a direct corollary of

Theorem 2.2. Under the hypotheses of Theorem 2.1 one has, for all $h \in L_2(\Omega)$,

$$\lim_{t\to+\infty} v^+(t,\cdot) = 0 \text{ in } L_2(\Omega) \tag{2.13}$$

and hence

$$\lim_{t\to+\infty} \| v(t,\cdot) - j(\cdot)\, v_0^+(t,\cdot)\|_{L_2(\Omega)} = 0 . \tag{2.14}$$

Proof of Theorem 2.2. Equation (2.14) can be written

$$\lim_{t\to+\infty} \left\| \left(e^{-itA_p^{1/2}} - J^* e^{-itA_{0,p}^{1/2}} \Phi_{0,p}^* \Phi_{-,p} \right) h \right\|_{L_2(\Omega)} = 0 \tag{2.15}$$

where $J^* : L_2(B_0) \to L_2(\Omega)$, the adjoint of the operator J defined in [5, (5.22)], is given by $J^* h(X) = j(y)\, h(X)|_\Omega$. Now the family of operators appearing in (2.15) is uniformly bounded for all $t \in R$. Hence to prove that (2.15) holds for all $h \in L_2(\Omega)$ it will suffice to verify it for all h from a dense subset of $L_2(\Omega)$. It will be convenient to use the dense subset $\mathcal{D}_0^- = \Phi_{-,p}^* \mathcal{D}_0$ where

$$\mathcal{D}_0 \subset \sum_{m\in Z} \oplus L_2(R_0) \tag{2.16}$$

is the set of all $g(q) = \{g_m(q) : m \in Z\}$ such that there is an $M = M(g)$ with the properties

$$g_m(q) \equiv 0 \text{ for } |m| > M \text{ , and} \tag{2.17}$$

$$g_m \in C_0^\infty(R_0 - \mathcal{E}_{m,p}) \text{ for } |m| \leq M \tag{2.18}$$

where $\mathcal{E}_{m,p}$ is the exceptional set of [5, (5.15)]. Moreover, it will suffice to verify (2.15) for functions of the form

$$h(X) = \int_0^\infty \phi_-(X,p+m,q)\ g(q)\ dq \tag{2.19}$$

where m is fixed and $g \in C_0^\infty(R_0 - \mathcal{E}_{m,p})$ has support in an interval $I \subset R_0 - \mathcal{E}_{m,p}$, since the case of a general $h \in \mathcal{D}_0^-$ then follows by superposition. Thus the proof of Theorem 2.2 may be completed by showing that if

$$v^+(t,X) = \int_I \phi'_-(X,p+m,q)\ e^{-it\omega(p+m,q)}\ g(q)\ dq \tag{2.20}$$

where $g \in C_0^\infty(R_0 - \mathcal{E}_{m,p})$ and supp $g \subset I$ then (2.13) holds.

The definition of the function $\phi'_-(X,p+m,q)$ [5, (5.4), (5.5), (5.13), (5.14)], together with [5, Theorem 4.15], implies that for fixed $m \in Z$ one has

$$\phi'_- \in C(\Omega \times (R_0^2 - \mathcal{E})) \tag{2.21}$$

where $\mathcal{E}$ is the exceptional set of [5, (2.30)]. Moreover, the far-field form of ϕ'_- is

$$\phi_-'(X,p+m,q) = \sum_{\ell\in L} a_\ell^-(p+m,q)\, e^{i(xp_\ell - yq_\ell)} + \rho_-(X,p+m,q) \tag{2.22}$$

where L is a finite set, independent of $q \in I$ (see [b, (9.33)ff] for the notation). Note that

$$a_\ell^-(p+m,q)\, e^{-iyq_\ell} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix(p+\ell)}\, \phi_-'(x,y,p+m,q)\, dx \ . \tag{2.23}$$

It follows from (2.21) and (2.23) that

$$a_\ell^-(\cdot+m,\cdot) \in C(R_0^2 - E) \ . \tag{2.24}$$

Moreover, by [6, Lemma 9.3] there exists a constant $\mu = \mu(p,m) > 0$ and for each $r' > r > h$ a constant $C = C(I,p,m,r,r')$ such that, for p and m fixed,

$$|\rho_-(X,p+m,q)| \le C\, e^{-\mu y} \text{ for all } X \in \Omega_r, \text{ and } q \in I \ . \tag{2.25}$$

Substitution of (2.22) in (2.20) gives

$$v^+(t,X) = v_1^+(t,X) + v_2^+(t,X) \tag{2.26}$$

where

$$v_1^+(t,X) = \sum_{\ell\in L} \left(\int_I a_\ell^-(p+m,q)\, e^{-i(yq_\ell + t\omega(p+m,q))}\, g(q)\, dq\right) e^{i(p+\ell)x} \tag{2.27}$$

Note that, by (2.24), each $a_\ell^-(p+m,\cdot)$ is continuous on the closed interval I. Now each of the integrals in (2.27) has the form of a modal wave in a simple waveguide; cf. [6, (9.63)ff]. It follows from [6, (9.69)] applied to the finite sum in (2.27) that

$$\lim_{t\to+\infty} \|v_1^+(t,\cdot)\|_{L_2(B_0)} = 0 . \tag{2.28}$$

It remains to show that $v_2^+(t,\cdot) \to 0$ in $L_2(\Omega)$ when $t \to +\infty$. This will be done by applying [6, Lemma 9.5] to $u = v_2^+ = v^+ - v_1^+$. To this end note that for all $t \in R$ one has $v^+(t,\cdot) \in L_2(\Omega)$ by (2.10) and $v_1^+(t,\cdot) \in L_2(B_0)$ by [6, (9.68)]. Thus $v_+^2(t,\cdot) = v^+(t,\cdot) - v_1^+(t,\cdot) \in L_2(\Omega)$ for all $t \in R$ which verifies [6, (9.75)].

The local decay property [6, (9.76)] follows from the local compactness property of the grating domain G, assumed in [5, §1], and the abstract decay theorem of [2]. The proof is the same as that of [3, Theorem 5.5] and is therefore not repeated here.

Finally, [6, (9.77)] follows directly from the estimate (2.25) and the representation

$$v_2^+(t,X) = \int_I \rho_-(X,p+m,q)\, e^{-it\omega(p+m,q)}\, g(q)\, dq \tag{2.29}$$

which imply

$$|v_2^+(t,X)| \le C\, e^{-\mu y} \int_I |g(q)|\, dq \text{ for all } X \in \Omega_r, \text{ and } t \in R . \tag{2.30}$$

This completes the proof of Theorem 2.2.

<u>Proof of Theorem 2.1</u>. The proof follows that of [3, Corollary 7.2]. In fact, the calculation given there, adapted to the present problem, gives the estimate

$$(2.31)\qquad \left\| \left(J_\Omega\, e^{-itA_p^{1/2}} - e^{-itA_{0,p}^{1/2}}\, \Phi_{0,p}^*\, \Phi_{-,p} \right) h \right\|_{L_2(B_0)}$$

$$\leq \left\| \left(e^{-itA_p^{1/2}} - J_\Omega^*\, e^{-itA_{0,p}^{1/2}}\, \Phi_{0,p}^*\, \Phi_{-,p} \right) h \right\|_{L_2(\Omega)}$$

$$+ \left\| e^{-itA_{0,p}^{1/2}} (\Phi_{0,p}^*\, \Phi_{-,p}\, h) \right\|_{L_2(B_{0,r})} + \left\| e^{-itA_p^{1/2}}\, h \right\|_{L_2(\Omega_{0,r})} .$$

The first term on the right in (2.31) tends to zero when $t \to +\infty$ by (2.15). The last two terms tend to zero when $t \to +\infty$ by the local decay property used in the proof of Theorem 2.2. It follows that the left-hand side of (2.31) tends to zero when $t \to +\infty$ which proves the existence of $W_{\pm,p}$ and equation (2.3). Finally, to verify (2.4) note that it can be written

$$(2.32)\qquad \Pi_p(\lambda) = \Phi_{\pm,p}^*\, \Phi_{0,p}\, \Pi_{0,p}(\lambda)\, \Phi_{0,p}^*\, \Phi_{\pm,p} \quad \text{for } \lambda \in R$$

by (2.3). The unitarity of $\Phi_{\pm,p}$ implies that an equivalent relation is

$$(2.33)\qquad \Phi_{\pm,p}\, \Pi_p(\lambda)\, \Phi_{\pm,p}^* = \Phi_{0,p}\, \Pi_{0,p}(\lambda)\, \Phi_{0,p}^* \quad \text{for } \lambda \in R \ .$$

But this last equation is correct because the two sides coincide with the operation

$$(2.34)\qquad \{g_m(q)\} \to \{H(\lambda - \omega^2(p+m,q))\, g_m(q)\}$$

in $\Sigma \oplus L_2(R_0)$; see [5, Theorem 5.7]. This completes the proof.

§3. Construction of the Wave Operators for A and A_0.

The notation of [5, §6] is used in this section. The purpose of the section is to prove the existence and completeness of the wave operators

$$(3.1) \qquad W_\pm = W_\pm(A_0^{1/2}, A^{1/2}, J_G) = \underset{t\to\pm\infty}{\text{s-lim}}\ e^{itA_0^{1/2}}\ J_G\ e^{-itA^{1/2}}$$

where $J_G : L_2(G) \to L_2(R_0^2)$ is defined by

$$(3.2) \qquad J_G\ h(X) = \begin{cases} h(X)\ , & X \in G\ , \\ 0\ , & X \in R_0^2 - G\ . \end{cases}$$

The principal results of the section are formulated as

Theorem 3.1. Let G be a grating domain [5, §1] and let $A = A(G)$ admit no surface waves. Then W_+ and W_- exist and are given by

$$(3.3) \qquad W_\pm = \Phi_0^*\ \Phi_\mp\ .$$

In particular, $W_\pm : L_2(G) \to L_2(R_0^2)$ are unitary operators and one has

$$(3.4) \qquad \Pi(\lambda) = W_\pm^*\ \Pi_0(\lambda)\ W_\pm \text{ for all } \lambda \in R\ .$$

The proof of Theorem 3.1 will be based on Theorem 2.1 and a series of lemmas that relate the grating propagators A and A_0 to the corresponding families of reduced propagators A_p and $A_{0,p}$, $-1/2 < p \le 1/2$.

The Mapping U. As a first step, the correspondence introduced in [5, (6.10)] will be extended to a unitary mapping $U : L_2(G) \to L_2((-1/2,1/2], L_2(\Omega))$. To see how this may be done note that if $f \in L_2(G)$ then

$$f(x+2\pi\ell,y)|_{\Omega} \in L_2(\Omega) \text{ for all } \ell \in Z, \text{ and} \tag{3.5}$$

$$\sum_{\ell\in Z} \|f(\cdot+2\pi\ell,\cdot)\|^2_{L_2(\Omega)} = \|f\|^2_{L_2(G)} < \infty . \tag{3.6}$$

Hence the Plancherel theory in the Lebesgue space $L_2((-1/2,1/2],L_2(\Omega))$ implies that the Fourier series

$$F(x,y,p) = \sum_{\ell\in Z} e^{-2\pi i\ell p} f(x+2\pi\ell,y)|_{\Omega} \tag{3.7}$$

converges in this space and Parseval's relation is valid. Combining this result and (3.6) gives

$$\|F\|_{L_2((-1/2,1/2],L_2(\Omega))} = \|f\|_{L_2(G)} \tag{3.8}$$

for all $f \in L_2(G)$.

<u>Lemma 3.2</u>. The mapping $U : L_2(G) \to L_2((-1/2,1/2],L_2(\Omega))$ defined by $Uf = F$ and (3.7) is unitary.

<u>Proof</u>. The preceding discussion implies that Uf is defined for all $f \in L_2(G)$ and U is isometric. The surjectivity of U follows from the Plancherel theory. Indeed, every $F \in L_2((-1/2,1/2],L_2(\Omega))$ has a Fourier development

$$F(X,p) = \sum_{\ell\in Z} e^{-2\pi i\ell p} F_{\ell}(X) , \tag{3.9}$$

convergent in $L_2((-1/2,1/2],L_2(\Omega))$. The Fourier coefficients in (3.9) are defined by the Bochner integrals

$$(3.10) \qquad F_\ell(X) = \int_{-1/2}^{1/2} e^{2\pi i \ell p} F(X,p)\, dp \in L_2(\Omega)$$

and Parseval's relation holds:

$$(3.11) \qquad \|F\|^2_{L_2((-1/2,1/2],L_2(\Omega))} = \sum_{\ell \in Z} \|F_\ell\|^2_{L_2(\Omega)} .$$

Thus to construct $f = U^{-1} F$ one need only require that $f(x+2\pi\ell,y)|_\Omega = F_\ell(X) \in L_2(\Omega)$, or

$$(3.12) \qquad f(X)|_{\Omega(\ell)} = F_\ell(x-2\pi\ell,y) \text{ for all } \ell \in Z .$$

Parseval's relation then guarantees that $f \in L_2(G)$ and (3.12) implies that $Uf = F$.

The next lemma makes it possible to construct operators $\Psi(A)$ from the corresponding reduced operators $\Psi(A_p)$ with $p \in (-1/2,1/2]$.

Lemma 3.3. For all bounded Borel functions $\Psi(\lambda)$ defined for $\lambda \geq 0$ and for all $f \in L_2(G)$ one has

$$(3.13) \qquad (U\, \Psi(A)f)(\cdot,p) = \Psi(A_p)\, Uf(\cdot,p) \in L_2(\Omega)$$

for almost every $p \in (-1/2,1/2]$.

Proof. The result will be derived from the R-B wave expansions for A and A_p and the corresponding Plancherel relations. To this end let $f,g \in L_2(G)$ and write $Uf = F$ and $Ug = G$. Moreover, assume that $\Phi_+ g = \hat{g}_+ \in L_2^{com}(R_0^2)$. Then Lemma 3.2 and the results of [5, §5 and §6] imply

$$(U\ \Psi(A)f,G) = (\Psi(A)f,g) = (\Phi_+\ \Psi(A)f,\ \Phi_+g) = \int_{R_0^2} \overline{\Psi(\omega^2(P))\ \hat{f}_+(P)}\ \hat{g}_+(P)dP$$

(3.14)

$$= \sum_{m\in Z} \int_{m-1/2}^{m+1/2} \int_0^\infty \overline{\Psi(\omega^2(p,q))\ \hat{f}_+(p,q)}\ \hat{g}_+(p,q)\ dq\ dp$$

$$= \sum_{m\in Z} \int_{-1/2}^{1/2} \int_0^\infty \overline{\Psi(\omega^2(p+m,q))\ \hat{f}_+(p+m,q)}\ \hat{g}_+(p+m,q)\ dq\ dp$$

$$= \int_{-1/2}^{1/2} \left(\sum_{m\in Z} \int_0^\infty \overline{\Psi(\omega^2(p+m,q))\ \tilde{F}_+(p+m,q,p)}\ \tilde{G}_+(p+m,q,p)dq \right) dp$$

$$= \int_{-1/2}^{1/2} (\Psi(A_p)\ F(\cdot,p),G(\cdot,p))_{L_2(\Omega)}\ dp\ .$$

Note that the hypothesis $\hat{g}_+ \in L_2^{com}(R_0^2)$ implies that the m-summation and interval of q-integration in (3.14) are finite. Moreover, since such functions $\hat{g}_+$ are dense in $L_2(R_0^2)$ the relation (3.14) holds for all $G \in L_2((-1/2,1/2],L_2(\Omega))$. On taking $G(X,p) = G_1(X)\ G_2(p)$ in (3.14) where $G_1 \in L_2(\Omega)$ and $G_2(p) \in L_2(-1/2,1/2]$ are arbitrary one gets (3.13).

The mapping U obviously depends on the grating domain $G : U = U_G$. In the special case $G = R_0^2$ let $U_0 = U_{R_0^2}$. With this notation one has

<u>Lemma 3.4</u>. The operators U, U_0, J_G and J_Ω satisfy

$$U_0\ J_G = J_\Omega\ U\ . \tag{3.15}$$

<u>Proof</u>. The definition of U implies that

$$(J_\Omega\ Uf)(X,p) = \sum_{\ell\in Z} e^{-2\pi i\ell p}\ J_\Omega\ (f(x+2\pi\ell,y)|_\Omega)\ , \tag{3.16}$$

$$(3.17) \qquad (U_0\, J_G\, f)(X,p) = \sum_{\ell\in Z} e^{-2\pi i\ell p}\, (J_G\, f(x+2\pi\ell,y)|_{B_0}),$$

for all $f \in L_2(G)$. These obviously define the same function, which implies (3.15).

The next lemma will be used to relate the wave operators for A and A_0 to those for A_p and $A_{0,p}$.

Lemma 3.5. For all $f \in L_2(G)$ one has

$$(3.18) \qquad U_0(\Phi_0^* \, \Phi_\pm\, f)(\cdot,p) = \Phi_{0,p}^* \, \Phi_{\pm,p}\, Uf(\cdot,p) \in L_2(B_0)$$

for almost every $p \in (-1/2,1/2]$.

Proof. The relation [5, (6.11)] can be written

$$(3.19) \qquad (\Phi_\pm f)(p+m,q) = (\Phi_{\pm,p}\, F(\cdot,p))_m(q) = (\Phi_{\pm,p}\, Uf(\cdot,p))_m(q)\ .$$

The relation was proved in [5] for all $f \in L_2^{com}(G)$. However (3.19), as a relationship in $\Sigma \oplus L_2(R_0)$, extends immediately from the dense set $L_2^{com}(G)$ to all of $L_2(G)$. In particular, specializing (3.19) to $G = R_0^2$ gives

$$(3.20) \qquad (\Phi_0\, f_0)(p+m,q) = (\Phi_{0,p}\, U_0\, f_0(\cdot,p))_m(q)$$

for all $f_0 \in L_2(R_0^2)$. Substituting $f_0 = \Phi_0^*\, \Phi_\pm\, f$ in (3.20) gives

$$(3.21) \qquad (\Phi_{0,p}(U_0\, \Phi_0^*\, \Phi_\pm f)(\cdot,p))_m(q) = \Phi_\pm f(p+m,q) = (\Phi_{\pm,p}\, Uf(\cdot,p))_m(q)$$

in $\Sigma \oplus L_2(R_0)$, by (3.19). Thus

$$(3.22) \qquad \Phi_{0,p}(U_0\, \Phi_0^*\, \Phi_\pm f)(\cdot,p) = \Phi_{\pm,p}\, Uf(\cdot,p)$$

for almost every $p \in (-1/2,1/2]$. (3.22) is equivalent to (3.18).

Lemmas 3.2-3.5 will now be shown to imply Theorem 3.1. The main step in the proof is described by

Theorem 3.6. For every $h \in L_2(G)$ and $H = Uh$ one has

$$
\begin{aligned}
&\left\| J_G\, e^{-itA^{1/2}}\, h - e^{-itA_0^{1/2}}\, \Phi_0^* \,\Phi_\pm\, h \right\|^2_{L_2(R_0^2)} \\
(3.23)\quad & \\
&= \int_{-1/2}^{1/2} \left\| J_\Omega\, e^{-itA_p^{1/2}} H(\cdot,p) - e^{-itA_{0,p}^{1/2}}\, \Phi_{0,p}^*\, \Phi_{\pm,p}\, H(\cdot,p) \right\|^2_{L_2(B_0)} dp\ .
\end{aligned}
$$

Proof. Lemma 3.2 implies that

$$
\begin{aligned}
&\left\| J_G\, e^{-itA^{1/2}}\, h - e^{-itA_0^{1/2}}\, \Phi_0^* \,\Phi_\pm\, h \right\|^2_{L_2(R_0^2)} \\
(3.24)\quad & \\
&= \int_{-1/2}^{1/2} \left\| U_0 (J_G\, e^{-itA^{1/2}}\, h - e^{-itA_0^{1/2}}\, \Phi_0\, \Phi_\pm h)(\cdot,p) \right\|^2_{L_2(B_0)} dp\ .
\end{aligned}
$$

Moreover, Lemmas 3.4 and 3.3 imply

$$
\begin{aligned}
&(U_0\, J_G\, e^{-itA^{1/2}}\, h)(\cdot,p) = (J_\Omega\, U\, e^{-itA^{1/2}}\, h)(\cdot,p) \\
(3.25)\quad & \\
&\qquad = J_\Omega\, e^{-itA_p^{1/2}}\, U\, h(\cdot,p)\ .
\end{aligned}
$$

Finally, Lemmas 3.3 and 3.5 imply

$$
\begin{aligned}
&(U_0\, e^{-itA_0^{1/2}}\, \Phi_0^*\, \Phi_\pm h)(\cdot,p) = e^{-itA_{0,p}^{1/2}}\, U_0(\Phi_0^*\, \Phi_\pm h)(\cdot,p) \\
(3.26)\quad & \\
&\qquad = e^{-itA_{0,p}^{1/2}}\, \Phi_{0,p}^*\, \Phi_{\pm,p}\, U\, h(\cdot,p)\ .
\end{aligned}
$$

Combining (3.24), (3.25) and (3.26) gives (3.23).

Proof of Theorem 3.1. Lemma 3.2 implies that $H(\cdot,p) \in L_2(\Omega)$ for almost every $p \in (-1/2,1/2]$. Hence the integrand on the right hand side

of (3.23) tends to zero when $t \to \mp\infty$ by Theorem 2.1 (see (2.31)). Moreover, the operators appearing in the integrand are all bounded with bound 1 and hence one has

$$\left\| J_\Omega e^{-itA_p^{1/2}} H(\cdot,p) - e^{-itA_{0,p}^{1/2}} \Phi_{0,p}^* \Phi_{\pm,p} H(\cdot,p) \right\|_{L_2(B_0)} \le 2 \| H(\cdot,p) \|_{L_2(\Omega)} \tag{3.27}$$

for all $t \in R$ and almost every $p \in (-1/2,1/2]$. Thus the existence of W_+ and W_-, and the relation (3.3), follow from (3.23) and Lebesgue's dominated convergence theorem. The final statement of Theorem 3.1, equation (3.4), follows from (3.3) and the eigenfunction expansions for A and A_0, exactly as in the proof of Theorem 2.1.

§4. Asymptotic Wave Functions and Energy Distributions.

In this section the existence of the wave operators $W_\pm$ is shown to imply that transient wave fields in grating domains G are asymptotically equal in the energy norm, for $t \to +\infty$, to transient wave fields in the degenerate grating domain R_0^2. The latter are then shown to be the restrictions to R_0^2 of free waves in R^2. Such free waves possess asymptotic wave functions in the sense of the author's monograph on scattering by bounded obstacles [3]. These results are shown below to imply that transient wave fields $u(t,X)$ with finite energy in grating domains possess asymptotic wave functions

$$(4.1) \qquad u_k^\infty(t,X) = r^{-1/2} F_k(r-t,\theta) \ , \ k = 0,1,2 \ ,$$

(where $X = (r \cos \theta, r \sin \theta)$) such that if $(t,x,y) = (X_0,X_1,X_2)$ and $D_k = \partial/\partial X_k$ for $k = 0,1,2$ then

$$(4.2) \qquad \lim_{t\to\infty} \| D_k u(t,\cdot) - u_k^\infty(t,\cdot) \|_{L_2(G)} = 0, \ k = 0,1,2 \ .$$

Moreover, the waveforms $F_k(\tau,\theta)$ are calculated from the initial state $f(X)$, $g(X)$ of $u(t,X)$. Finally, (4.2) and the results of [3, Ch. 8] are used to calculate the asymptotic distributions of energy for transient wave fields in grating domains.

The starting point for the calculation of the asymptotic wave functions (4.1) is the complex-valued wave function $v(t,X)$ defined by (1.12), (1.13). The existence of the wave operator W_+ defined by (3.1) implies that

$$(4.3) \qquad e^{-itA^{1/2}} h \sim e^{-itA_0^{1/2}} W_+ h \ , \ t \to +\infty \ ,$$

in the sense of convergence in $L_2(G)$. Moreover, if $h \in D(A^{1/2})$ then the analogue of (4.3) holds for the first derivatives. This result may be formulated as a generalization of the corresponding result for exterior domains [3, Theorem 7.5], as follows.

Theorem 4.1. Let G satisfy the hypotheses of Theorem 3.1 and let $h \in D(A^{1/2})$. Then $v(t,\cdot) = e^{-itA^{1/2}} h$ is a solution wFE in G, $h_0^+ = W_+ h \in D(A_0^{1/2})$, $v_0(t,\cdot) = e^{-itA_0^{1/2}} h^+$ *is a solution wFE in* R_0^2 *and*

$$\lim_{t\to+\infty} \| D_k v(t,\cdot) - D_k v_0(t,\cdot)\|_{L_2(G)} = 0 \text{ for } k = 0,1,2 \,. \tag{4.4}$$

The proof is precisely the same as the one for exterior domains given in [3] and is therefore omitted.

The initial state $h_0^+ = W_+ h$ for the wave field $v_0(t,X)$ satisfies

$$\hat{h}_0^+ = \Phi_0 \, h_0^+ = \Phi_- \, h = \hat{h}_- \tag{4.5}$$

by (3.3). Thus v_0 has the R-B wave representation (see (1.14))

$$v_0(t,X) = \int_{R_0^2} \psi_0(X,P) \; e^{-it\omega(P)} \; \hat{h}_-(P) \, dP \,. \tag{4.6}$$

To show that $v_0(t,X)$ has a continuation to a wave field wFE in R^2 the Neumann and Dirichlet cases will be treated separately.

The Neumann Case. Here one has

$$\psi_0(X,P) = \psi_0^N(X,P) = \frac{1}{2\pi} \, e^{ipx}(e^{iqy} + e^{-iqy}) \tag{4.7}$$

and substitution in (4.6) gives, after a simple transformation,

$$v_0(t,X) = \frac{1}{2\pi} \int_{R^2} e^{i(xp+yq-t\omega(p,q))} \; \hat{h}_0(p,q) \, dpdq \tag{4.8}$$

where

$$(4.9)\qquad \hat{h}_0(p,q) = \begin{cases} \hat{h}_-(p,q)\ , & (p,q) \in R_0^2\ , \\ \hat{h}_-(p,-q), & (p,-q) \in R_0^2\ . \end{cases}$$

The Dirichlet Case. Here if ψ_0 is normalized by

$$(4.10)\qquad \psi_0(X,P) = i\ \psi_0^D(X,P) = \frac{1}{2\pi} e^{ipx} (e^{iqy} - e^{-iqy})$$

then substitution in (4.6) again gives (4.8), but with

$$(4.11)\qquad \hat{h}_0(p,q) = \begin{cases} \hat{h}_-(p,q)\ , & (p,q) \in R_0^2\ , \\ -\hat{h}_-(p,-q), & (p,-q) \in R_0^2\ . \end{cases}$$

Thus in both cases $v_0(t,X)$ has a continuation (4.8) to a wave field in R^2. Moreover, the hypothesis $h \in D(A^{1/2})$ of Theorem 4.1 implies that $\hat{h}_-(p,q)$ and $\sqrt{p^2+q^2}\ \hat{h}_-(p,q)$ are in $L_2(R_0^2)$ and hence $\hat{h}_0(p,q)$ and $\sqrt{p^2+q^2}\ \hat{h}_0(p,q)$ are in $L_2(R^2)$. It follows that the extended wave field (4.8) is a solution wFE in R^2. Thus the results of [3, Ch. 2] are applicable and allow the construction of asymptotic wave functions

$$(4.12)\qquad v_k^\infty(t,X) = r^{-1/2} H_k(r-t,\theta)\ ,\ k = 0,1,2\ ,$$

such that

$$(4.13)\qquad \lim_{t\to+\infty} \| D_k\ v_0(t,\cdot) - v_k^\infty(t,\cdot)\|_{L_2(R^2)} = 0,\ k = 0,1,2\ .$$

By restricting the functions to G one obtains

Corollary 4.2. Under the hypotheses of Theorem 4.1 one has

$$(4.14)\qquad \lim_{t\to+\infty} \| D_k v(t,\cdot) - v_k^\infty(t,\cdot)\|_{L_2(G)} = 0,\ k = 0,1,2\ ,$$

where the function v_k^∞ are given by (4.12) with waveforms H_k defined by

$$H_0(\tau,\theta) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega}\, \hat{h}_-(\omega\cos\theta, \omega\sin\theta)(-i\omega)^{1/2}\, d\omega \,, \tag{4.15}$$

convergent in $L_2(R \times [0,\pi])$, and

$$H_1(\tau,\theta) = -H_0(\tau,\theta)\cos\theta \,,\; H_2(\tau,\theta) = -H_0(\tau,\theta)\sin\theta \,. \tag{4.16}$$

Equations (4.14) follow from (4.4) and (4.13) by the triangle inequality. Equations (4.13), (4.15) and (4.16) follow directly from [3, Ch. 2]; see the proof of [3, Theorem 2.10].

To obtain corresponding results for the real-valued wave field $u(t,X)$ generated by the initial state f,g one need only take the real part of $v(t,X)$ and use equation (1.13) which relates h to f and g. This leads to

Theorem 4.3. Let G satisfy the hypotheses of Theorem 3.1. Let $f \in D(A^{1/2})$ and $g \in L_2(G)$ and define asymptotic wave functions

$$u_k^\infty(t,X) = r^{-1/2}\, F_k(r-t,\theta) \,,\; k = 0,1,2 \,, \tag{4.17}$$

by

$$F_0(\tau,\theta) = \mathrm{Re}\left\{\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega}[\hat{g}_-(\omega\cos\theta,\omega\sin\theta) - i\omega\hat{f}_-(\omega\cos\theta,\omega\sin\theta)](-i\omega)^{1/2}\, d\omega\right\} \,, \tag{4.18}$$

convergent in $L_2(R \times [0,\pi])$, and

$$F_1(\tau,\theta) = -F_0(\tau,\theta)\cos\theta \,,\; F_2(\tau,\theta) = -F_0(\tau,\theta)\sin\theta) \,. \tag{4.19}$$

Then the solution wFE (1.10) generated by f and g satisfies

$$\lim_{t\to+\infty} \| D_k\, u(t,\cdot) - u_k^\infty(t,\cdot)\|_{L_2(G)} = 0,\ k = 0,1,2 \ . \tag{4.20}$$

Proof. To begin, assume that $g \in D(A^{-1/2})$ and define h by (1.13). Then $h \in D(A^{1/2})$ and Corollary 4.2 is applicable to $v(t,\cdot) = e^{-itA^{1/2}} h$. Moreover,

$$|P|\ \hat{h}_-(P) = |P|\ \hat{f}_-(P) + i\ \hat{g}_-(P) \tag{4.21}$$

which implies that $F_k = \mathrm{Re}\ H_k$ and hence

$$u_k^\infty(t,X) = \mathrm{Re}\ \{v_k^\infty(t,X)\}\ ,\ k = 0,1,2\ . \tag{4.22}$$

Thus (4.20) follows from (4.14) by the triangle inequality. To remove the restriction that $g \in D(A^{-1/2})$ note that $D(A^{-1/2})$ is dense in $L_2(G)$, by the spectral theorem. Moreover, one has (see [3, Theorem 2.5])

$$\begin{aligned} \|u_0^\infty(t,\cdot)\|_{L_2(G)} &\le \|v_0^\infty(t,\cdot)\|_{L_2(R_0^2)} \le \|H_0\| = \|\hat{H}_0\| \\ &= \left(\int_0^\infty \int_0^\pi \omega^3\ |\hat{h}_-(\omega\cos\theta, \omega\sin\theta)|^2\ d\theta\ d\omega\right)^{1/2} \\ &= \left(\iint_{R_0^2} |P|^2\ |\hat{h}_-(P)|^2\ dP\right)^{1/2} = \|\,|P|\ \hat{h}_-(P)\|_{L_2(R_0^2)} \\ &\le \|\,|P|\hat{f}_-(P)\| + \|\hat{g}_-\| = \|A^{1/2} f\| + \|g\|\ . \end{aligned} \tag{4.23}$$

Similarly,

$$\|u_k^\infty(t,\cdot)\|_{L_2(G)} \le \|u_0^\infty(t,\cdot)\| \le \|A^{1/2} f\| + \|g\|\ ,\ k = 1,2\ . \tag{4.24}$$

Finally, the conservation of energy theorem implies that

$$\|D_k u(t,\cdot)\|_{L_2(G)} \le E(u,G,t)^{1/2} = E(u,G,0)^{1/2}$$

(4.25)

$$= (\|A^{1/2} f\|^2 + \|g\|^2)^{1/2} \le \|A^{1/2} f\| + \|g\| \ , \ k = 0,1,2 \ .$$

It follows from (4.23), (4.24) and (4.25) that (4.20) can be extended to all $f \in D(A^{1/2})$ and $g \in L_2(G)$ by a well-known density argument (see, e.g., [3, Proof of Theorem 2.6]).

Theorem 4.3 permits the extension to grating domains of the results on asymptotic energy distributions in exterior domains given in [3]. The principal results are formulated below. The proofs are identical to those of [3] and are therefore omitted.

Corollary 4.4 (Scattering into Cones). Let

(4.26) $$\Gamma = \{X = (r \cos \theta, r \sin \theta) : r > 0 \text{ and } \theta \in \Gamma_0\}$$

where Γ_0 is a Lebesgue-measurable subset of $[0,\pi]$, and let $X_0 \in R^2$. Then under the hypotheses of Theorem 4.3 the limit

(4.27) $$E^{\infty}(u, G \cap (\Gamma + X_0)) = \lim_{t \to +\infty} E(u, G \cap (\Gamma + X_0), t)$$

exists and

(4.28) $$E^{\infty}(u, G \cap (\Gamma + X_0)) = \int_{\Gamma} \left| |P| \ \hat{f}_-(P) + i \ \hat{g}_-(P) \right|^2 dP \ .$$

Corollary 4.5 (Transiency of Energy in Slabs). Let

(4.29) $$\Sigma = \{X : d_1 \le X \cdot X_0 \le d_2\}$$

where d_1 and d_2 $(> d_1)$ are constants and $X_0 \in R^2$ is a unit vector. Then under the hypotheses of Theorem 4.3 one has

$$\lim_{t\to+\infty} E(u, G \cap \Sigma, t) = 0 . \tag{4.30}$$

Note that Corollary 4.5 implies the transiency of the energy in bounded sets since every bounded set $K \subset R^2$ is contained in a slab (4.29).

§5. Construction and Structure of the S-Matrix.

The scattering operator associated with the pair A, A_0 is the linear operator $S : L_2(R_0^2) \to L_2(R_0^2)$ defined by

$$S = W_+ W_-^* . \tag{5.1}$$

The corresponding operator in $L_2(R_0^2)$ defined by

$$\hat{S} = \Phi_0 \, S \, \Phi_0^* \tag{5.2}$$

is the Heisenberg operator, or S-matrix, for the pair A, A_0. From the representation $W_\pm = \Phi_0^* \Phi_\mp$ of Theorem 3.1 one has

$$\hat{S} = \Phi_- \Phi_+^* . \tag{5.3}$$

The unitarity of $W_\pm$ and Φ_0 imply that S and $\hat{S}$ are unitary operators in $L_2(R_0^2)$. The purpose of this section is to calculate $\hat{S}$. Specifically, it will be shown how $\hat{S}$ can be constructed from the scattering coefficients $\{c_\ell^\pm(p,q)\}$ of the R-B waves $\psi_\pm(x,y,p,q)$ and the relationships among these coefficients imposed by the unitarity of S will be determined. The role of the S-matrix in the scattering of transient fields by gratings will be developed in §6.

If $h \in L_2(G)$ then (5.3) implies that the functions $\hat{h}_+ = \Phi_+ h$ and $\hat{h}_- = \Phi_- h$ satisfy

$$\hat{h}_- = \hat{S} \, \hat{h}_+ . \tag{5.4}$$

Thus $\hat{S}$ may be calculated by calculating the relationship between $\hat{h}_-$ and $\hat{h}_+$. This will be done by using the incoming and outgoing R-B wave

representations of $v(t,\cdot) = e^{-itA^{1/2}} h$ to calculate in two different ways the asymptotic wave function in $L_2(G)$ associated with $v(t,\cdot)$; say

$$v^{\infty}(t,X) = r^{-1/2} H(r-t,\theta) \quad (5.5)$$

The function $H \in L_2(R \times [0,\pi])$ is uniquely determined by the condition

$$\lim_{t\to+\infty} \| v(t,\cdot) - v^{\infty}(t,\cdot) \|_{L_2(G)} = 0 \; ; \quad (5.6)$$

see [3, Theorem 2.5]. The equality of the representations of H obtained from the incoming and outgoing representations of $v(t,\cdot)$ provides the required relationship between $\hat{h}_-$ and $\hat{h}_+$.

<u>First Calculation of H</u>. Theorem 3.1 implies that

$$\lim_{t\to+\infty} \| v(t,\cdot) - v_0(t,\cdot) \|_{L_2(G)} = 0 \quad (5.7)$$

where $v_0(t,\cdot) = e^{-itA_0^{1/2}} h_0$ is the wave function in $L_2(R_0^2)$ of Theorem 4.1. Proceeding as in the proof of Corollary 4.2 one shows that (5.5), (5.6) hold with

$$H(\tau,\theta) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} e^{i\tau\omega} \, \hat{h}_-(\omega \cos\theta, \omega \sin\theta)(-i\omega)^{1/2} \, d\omega \, . \quad (5.8)$$

The convergence $v_0(t,\cdot) - v^{\infty}(t,\cdot) \to 0$ in $L_2(R^2)$ was proved in [3, Theorem 2.6].

<u>A Classification of the R-B Waves</u>. The second calculation of H will be based on the outgoing representation

$$v(t,X) = \text{l.i.m.} \int_{R_0^2} \psi_+(X,P) \, e^{-it\omega(P)} \, \hat{h}_+(P) \, dP \, . \quad (5.9)$$

The R-B wave ψ_+ has the expansion for $y \geq h$ [5, (2.26)]

$$\psi_+(x,y,p,q) = (2\pi)^{-1}\, e^{i(px-qy)} + \sum_{(p+\ell)^2<p^2+q^2} c_\ell^+(p,q)\, e^{i(p_\ell x+q_\ell y)}$$

(5.10)

$$+ \sum_{(p+\ell)^2\geq p^2+q^2} c_\ell^+(p,q)\, e^{ip_\ell x}\, e^{-\{(p+\ell)^2-p^2-q^2\}^{1/2}\, y}$$

where

(5.11) $$(p_\ell, q_\ell) = (p + \ell, \{p^2 + q^2 - (p+\ell)^2\}^{1/2}) .$$

The first sum in (5.10) is a superposition of a finite number of outgoing plane waves, while the second sum is an exponentially decreasing function of y for $(x,y) \in R_0^2 - E$ [6, Lemma 9.3]. In the calculation of the asymptotic wave function (5.5) from (5.9) and (5.10) a difficulty arises because the number of terms in the first sum varies with $(p,q) \in R_0^2$. This number changes at the points $(p,q) \in E$ and is constant on the components of the set $R_0^2 - E$. It will therefore be convenient to classify the R-B waves by means of these components. Note that $(p,q) \in E$ if and only if $q > 0$ and

(5.12) $$q_\ell^2 \equiv p^2 + q^2 - (p+\ell)^2 = 0 \ , \ \ell \in Z - \{0\} .$$

The set π_ℓ so defined is the portion lying in R_0^2 of the parabola with focus at (0,0) and vertex at $(-\ell/2, 0)$. The curves π_ℓ and π_m are disjoint if $\ell m > 0$ and intersect orthogonally if $\ell m < 0$. Thus if

$$\mathcal{O}_m = R_0^2 \cap \{(p,q) : |p + m| < \sqrt{p^2 + q^2} < |p + m + 1|\} ,$$

(5.13)

$$\mathcal{O}_{-n} = R_0^2 \cap \{(p,q) : |p - n| < \sqrt{p^2 + q^2} < |p - n - 1|\} ,$$

where $m,n = 0,1,2,\cdots$, then $\mathcal{O}_m$ is the domain between π_m and π_{m+1}, $\mathcal{O}_{-n}$ is the domain between π_{-n} and π_{-n-1} and the sets

$$(5.14) \qquad \mathcal{O}_{m,n} = \mathcal{O}_m \cap \mathcal{O}_{-n} , \quad m,n = 0,1,2,\cdots,$$

are the components of $R_0^2 - \mathcal{E}$:

$$(5.15) \qquad R_0^2 - \mathcal{E} = \bigcup_{m,n=0}^{\infty} \mathcal{O}_{m,n} .$$

Note that $(p,q) \in \mathcal{O}_{m,n}$ if and only if the expansion (5.10) of $\psi_+(x,y,p,q)$ contains exactly $m + n + 1$ outgoing plane waves with the propagation directions (p_ℓ,q_ℓ), $-n \le \ell \le m$.

<u>Second Calculation of H</u>. In calculating $\hat{S}$ it will suffice to determine $\hat{S}\,\hat{h}_+$ for functions $\hat{h}_+$ of a dense set in $L_2(R_0^2)$ because $\hat{S}$ is known to be unitary. For this purpose it will be convenient to use functions $\hat{h}_+ \in C_0^\infty(R_0^2 - \mathcal{E})$. For such functions, supp $\hat{h}_+$ is a compact subset of the set (5.15). Hence, supp $\hat{h}_+$ meets only finitely many of the sets $\mathcal{O}_{m,n}$ and each component of supp $\hat{h}_+$ lies in one of these sets. Thus in calculating $\hat{S}\,\hat{h}_+$ it will be enough to consider the case where

$$(5.16) \qquad \text{supp } \hat{h}_+ = K \subset \mathcal{O}_{m,n} , \ m \text{ and } n \text{ fixed} .$$

The case of a general $\hat{h}_+ \in C_0^\infty(R_0^2 - \mathcal{E})$ may then be obtained by superposition. With this hypothesis the wave function (5.9) becomes

$$(5.17) \qquad v(t,X) = \int_K \psi_+(X,P)\, e^{-it\omega(P)}\, \hat{h}_+(P)\, dP .$$

The asymptotic wave function (5.5) for $v(t,X)$ will be calculated from (5.17) by substituting the expansion (5.10) and determining the

behavior for $t \to +\infty$ of the terms in the resulting sum. For this purpose a bound is needed for the remainder in (5.10) that is uniform in $(p,q) \in K$. Thus a refinement of [6, Lemma 9.3] is needed since the latter is valid for fixed p only. The following generalization of [6, Lemma 9.3] will be proved.

Lemma 5.1. Define the remainder $\sigma_\pm(X,p,q)$ for all $X \in G$ and $(p,q) \in \mathcal{O}_{m,n}$ by

$$(5.18)\quad \psi_\pm(X,p,q) = (2\pi)^{-1}\, e^{i(px\mp qy)} + \sum_{\ell=-n}^{m} c_\ell^\pm(p,q)\, e^{i(p_\ell x\pm q_\ell y)} + \sigma_\pm(X,p,q)\ .$$

Then for each compact set $K \subset \mathcal{O}_{m,n}$ and each $r' > r > h$ there exist constants $\mu = \mu(K) > 0$ and $C = C(K,r',r)$ such that

$$(5.19)\qquad |\sigma_\pm(X,p,q)| \le C\, e^{-\mu y} \text{ for all } X \in R_r^2, \text{ and } (p,q) \in K\ .$$

Proof. Only the case of σ_+ will be discussed since the other case then follows from the relation [5, (2.25)]. The proof will parallel that of [6, Lemma 9.3]. Note that [5, (6.5)] implies that

$$(5.20)\qquad \sigma_+(x,y,p,q) = e^{2\pi i\ell p}\, \rho_+(x - 2\pi\ell,y,p,q)\ ,\ (x,y) \in \Omega^{(\ell)}\ ,$$

where ρ_+ is defined by [6, (9.39)] with

$$(5.21)\qquad L' = \{\ell \in Z : \ell \le -n - 1 \text{ or } \ell \ge m + 1\}$$

for all $(p,q) \in \mathcal{O}_{m,n}$. Thus to prove (5.19) it is enough to show that

$$(5.22)\qquad |\rho_+(X,p,q)| \le C\, e^{-\mu y} \text{ for all } X \in \Omega_r, \text{ and } (p,q) \in K\ .$$

Proceeding as in [6, Proof of Lemma 9.3], one has

$$(5.23)\qquad |\phi'_{+\ell}(r,p,q)|^2 \le C_0^2(2\pi)^{-1}\|\phi'_+(\cdot,p,q)\|^2_{1;h,r} \text{ for all } \ell \in L'$$

where $C_0 = C_0(h,r)$. Now the right-hand side of (5.23) is a continuous function of $(p,q) \in R_0^2 - \mathcal{E}$ by [5, Theorem 6.1]. Thus there exists a $C_1 = C_1(K,r)$ such that

$$(5.24)\qquad |\phi'_{+\ell}(r,p,q)| \le C_1 \text{ for all } (p,q) \in K \text{ and } \ell \in L' .$$

Next, since K is a compact subset of $\mathcal{O}_{m,n}$ there exist constants $\mu_+ = \mu_+(K) > 0$ and $\mu_- = \mu_-(K) > 0$ such that

$$(5.25)\qquad \begin{aligned} &(p+\ell)^2 - p^2 - q^2 \ge \mu_+^2 \text{ for all } (p,q) \in K \text{ and } \ell \ge m+1 , \\ &(p+\ell)^2 - p^2 - q^2 \ge \mu_-^2 \text{ for all } (p,q) \in K \text{ and } \ell \le -n-1 . \end{aligned}$$

whence

$$(5.26)\qquad \{(p+\ell)^2 - p^2 - q^2\}^{1/2} \ge \mu(K) = \mathrm{Min}\,(\mu_+(K),\mu_-(K)) > 0$$

for all $(p,q) \in K$ and $\ell \in L'$. It follows that for all $X \in \Omega_r$, and $(p,q) \in K$ one has

$$(5.27)\qquad \begin{aligned} |\rho_+(X,p,q)| &\le \sum_{\ell\in L'} |\phi'_{+\ell}(y,p,q)| \\ &\le \sum_{\ell\in L'} |\phi'_{+\ell}(r,p,q)| \exp\{-(y-r)((p+\ell)^2-p^2-q^2)^{1/2}\} \\ &\le C_1 \sum_{\ell\in L'} \exp\{-(y-r)((p+\ell)^2-p^2-q^2)^{1/2}\} \end{aligned}$$

$$\leq C_1 \sum_{\ell \in L'} \exp\{-(y-r')((p+\ell)^2-p^2-q^2)^{1/2}\} \exp\{-(r'-r)((p+\ell)^2-p^2-q^2)^{1/2}\}$$

(5.27 cont.)

$$\leq C_1 e^{-(y-r')\mu(K)} \sum_{\ell \in L'} \exp\{-(r'-r)((p+\ell)^2-p^2-q^2)^{1/2}\} .$$

Now

$$(5.28) \qquad \Sigma(r'-r,p,q) = \sum_{\ell \in L'} \exp\{-(r'-r)((p+\ell)^2-p^2-q^2)^{1/2}\}$$

is a continuous function of $(p,q) \in \mathcal{O}_{m,n}$ and hence for each compact $K \subset \mathcal{O}_{m,n}$ there is a constant $M(r' - r,K)$ such that

$$(5.29) \qquad \Sigma(r'-r,p,q) \leq M(r'-r,K) \text{ for all } (p,q) \in K .$$

Combining (5.27), (5.28) and (5.29) gives (5.22) with

$$(5.30) \qquad C = C_1(K,r)\, e^{r'\mu(K)}\, M(r'-r,K) .$$

<u>Second Calculation of H (continued)</u>. Substitution of (5.18) into (5.17) gives the decomposition

$$(5.31) \quad v(t,X) = v^{in}(t,X) + \sum_{\ell=-n}^{m} v_\ell^{out}(t,X) + v_\sigma(t,X) , \quad t \in R , \; X \in G ,$$

where

$$(5.32) \qquad v^{in}(t,X) = \frac{1}{2\pi} \int_K e^{i(px-qy-t\omega(p,q))}\, \hat{h}_+(p,q)\, dpdq ,$$

$$(5.33) \quad v_\ell^{out}(t,X) = \int_K e^{i(p_\ell x+q_\ell y-t\omega(p,q))}\, c_\ell^+(p,q)\, \hat{h}_+(p,q)\, dpdq , \text{ and}$$

$$(5.34)\qquad v_\sigma(t,X) = \int_K e^{-it\omega(p,q)}\, \sigma_+(X,p,q)\, \hat{h}_+(p,q)\, dpdq\ .$$

Recall that by assumption $\hat{h}_+ \in C_0^\infty(R_0^2 - E)$ satisfies (5.16) and $c_\ell^+(p,q) \in C(R_0^2 - E)$ (see (2.23), (2.24)). Thus the integrands in the above integrals are all continuous. The second calculation of H will now be carried out by calculating the asymptotic wave function in $L_2(G)$ of each term on the right-hand side of (5.31).

<u>The Partial Wave $v^{in}(t,X)$</u>. The change of variables $(p',q') = (p,-q)$ in (5.32) gives

$$(5.35)\qquad v^{in}(t,X) = \frac{1}{2\pi}\int_{K'} e^{i(p'x+q'y-t\omega(p',q'))}\, \hat{h}_+(p',-q')\, dp'dq'$$

where $K' = \{(p',q') : (p,q) \in K\} \subset R^2 - R_0^2$. Thus $v^{in}(t,X)$ is a free wave in R^2 and hence has an asymptotic wave function $r^{-1/2}\, H^{in}(r - t,\theta)$ with waveform defined by [3, Theorem 2.6]

$$(5.36)\qquad H^{in}(\tau,\theta) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{i\tau\omega}\, \hat{h}_+(\omega\cos\theta, -\omega\sin\theta)(-i\omega)^{1/2}\, d\omega\ .$$

In particular, $H^{in}(\tau,\theta) \equiv 0$ for $0 \le \theta \le \pi$ because $K' = \text{supp}\ \hat{h}_+(p,-q) \subset R^2 - R_0^2$.

<u>The Partial Waves $v_\ell^{out}(t,X)$.</u> To interpret these terms let $\ell \in Z$ and consider the mapping X_ℓ defined by

$$(5.37)\qquad (p_\ell,q_\ell) = X_\ell(p,q) = (p + \ell, \{p^2 + q^2 - (p + \ell)^2\}^{1/2})\ .$$

X_ℓ is analytic on the domain

(5.38) $$D(X_\ell) = \{(p,q) : \sqrt{p^2+q^2} > |p+\ell|,\ q > 0\}$$

and maps it bijectively onto the range

(5.39) $$R(X_\ell) = \{(p_\ell,q_\ell) : \sqrt{p^2+q^2} > |p_\ell - \ell|\ ,\ q_\ell > 0\}\ .$$

Moreover,

(5.40) $$X_\ell^{-1} = X_{-\ell}\ ,\ \ell \in Z\ ,$$

and the Jacobian of X_ℓ is

(5.41) $$\frac{\partial(p_\ell,q_\ell)}{\partial(p,q)} = \frac{q}{q_\ell}\ .$$

Note that $\omega(p,q)$ is invariant under X_ℓ :

(5.42) $$\omega(p_\ell,q_\ell) = p_\ell^2 + q_\ell^2 = p^2 + q^2 = \omega(p,q)\ .$$

It is easy to verify that

(5.43) $$X_\ell\ \mathcal{O}_{m,n} = \mathcal{O}_{m-\ell,n+\ell} \text{ for } -n \le \ell \le m\ .$$

Hence the hypothesis $K \subset \mathcal{O}_{m,n}$ implies that

(5.44) $$X_\ell\ K \equiv K_\ell \subset \mathcal{O}_{m-\ell,n+\ell} \text{ for } -n \le \ell \le m\ .$$

In particular, one has

(5.45) $$K_j \cap K_\ell = \phi \text{ for } j \ne \ell\ .$$

On making the change of variables $(p,q) \rightarrow (p_\ell,q_\ell) = X_\ell(p,q)$ in (5.33) one finds the representation

$$(5.46)\quad v_\ell^{out}(t,X) = \int_{K_\ell} e^{i(xp_\ell+yq_\ell-t\omega(p_\ell,q_\ell))}\left\{c_\ell^{+}(p,q)\hat{h}_+(p,q)\ \frac{q_\ell}{q}\right\} dp_\ell\ dq_\ell$$

where $(p,q) = X_{-\ell}(p_\ell,q_\ell)$ in the integrand. Thus $v_\ell^{out}(t,X)$ is also a free wave in R^2 and has an asymptotic wave function $r^{-1/2}\,H_\ell(r-t,\theta)$ with waveform defined by

$$(5.47)\qquad H_\ell(\tau,\theta) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{i\tau\omega}\ \hat{H}_\ell(\omega,\theta)\ d\omega$$

and

$$(5.48)\quad \hat{H}_\ell(\omega,\theta) = 2\pi(-i\omega)^{1/2}\,\frac{q_\ell}{q}\ c_\ell^{+}(p,q)\ \hat{h}_+(p,q)\Big|_{(p,q)=X_{-\ell}(\omega\cos\theta,\omega\sin\theta)} .$$

A simple calculation shows that

$$(5.49)\qquad \|H_\ell\|^2_{L_2(R\times[0,\pi])} = (2\pi)^2\int_K |c_\ell^{+}(p,q)\ \hat{h}_+(p,q)|^2\ \frac{q_\ell}{q}\ dp\ dq$$

where q_ℓ is defined by (5.37). In particular, $H_\ell \in L_2(R\times[0,\pi])$ because the integrand in (5.49) is continuous on K.

The Partial Wave $v_\sigma(t,X)$. Equation (5.31) may be written

$$(5.50)\qquad v_\sigma(t,X) = v(t,X) - v^{in}(t,X) - \sum_{\ell=-n}^{m} v_\ell^{out}(t,X)$$

for all $t \in R$ and $X \in G$. Moreover, it has been shown that

$$(5.51)\qquad \begin{aligned} v(t,X) &= r^{-1/2}\,H(r-t,\theta) + o(1)\ , \\ v^{in}(t,X) &= o(1)\ , \\ v_\ell^{out}(t,X) &= r^{-1/2}\,H_\ell(r-t,\theta) + o(1) \end{aligned}$$

where each term $o(1) \in L_2(G)$ for all $t \in R$ and tends to zero in $L_2(G)$ when $t \to +\infty$. These results imply that

$$(5.52) \qquad v_\sigma(t,X) = r^{-1/2} H_\sigma(r-t,\theta) + o(1)$$

where $o(1) \to 0$ in $L_2(G)$ when $t \to +\infty$ and

$$(5.53) \qquad H_\sigma(\tau,\theta) = H(\tau,\theta) - \sum_{\ell=-n}^{m} H_\ell(\tau,\theta) \text{ in } L_2(R \times [0,\pi]) .$$

On the other hand (5.34) and Lemma 5.1 imply that

$$(5.54) \qquad |v_\sigma(t,X)| \le C_\sigma \, e^{-\mu y} \text{ for all } t \in R \text{ and } X \in R_r^2,$$

where

$$(5.55) \qquad C_\sigma = C \int_K |\hat{h}_+(p,q)| \, dp \, dq$$

and $\mu = \mu(K) > 0$ and $C = C(K,r',r)$ are the constants of the lemma. The second calculation of H will now be completed by showing that (5.52) and (5.54) imply

<u>Theorem 5.2</u>. $H_\sigma(\tau,\theta) \equiv 0$ and hence

$$(5.56) \qquad H(\tau,\theta) = \sum_{\ell=-n}^{m} H_\ell(\tau,\theta) \text{ in } L_2(R \times [0,\pi]) .$$

<u>Proof</u>. Let ε be an arbitrary number in the interval $0 < \varepsilon < \pi/2$ and consider the sector

$$(5.57) \qquad \Gamma_\varepsilon = \{(x,y) = (r\cos\theta, r\sin\theta) : \varepsilon < \theta < \pi - \varepsilon\} .$$

By (5.52), the local decay of asymptotic wave functions [3, p. 32] and the triangle inequality, one has

$$(5.58)\qquad \begin{aligned}\int_{G\cap\Gamma_\varepsilon} |v_\sigma(t,X)|^2\, dX &= \int_{\Gamma_\varepsilon} |H_\sigma(r-t,\theta)|^2\, r^{-1}\, dX + o(1)\\ &= \int_0^\infty \int_\varepsilon^{\pi-\varepsilon} |H_\sigma(r-t,\theta)|^2\, d\theta\, dr + o(1)\\ &= \int_{-t}^\infty \int_\varepsilon^{\pi-\varepsilon} |H_\sigma(\tau,\theta)|^2\, d\theta\, d\tau + o(1)\end{aligned}$$

where $o(1) \to 0$ when $t \to \infty$. Thus passage to the limit in (5.58) gives

$$(5.59)\qquad \lim_{t\to+\infty} \int_{G\cap\Gamma_\varepsilon} |v_\sigma(t,X)|^2\, dX = \int_{-\infty}^\infty \int_\varepsilon^{\pi-\varepsilon} |H_\sigma(\tau,\theta)|^2\, d\theta\, d\tau\ .$$

On the other hand, writing $R^2_{a,b} = \{(x,y) : x \in R,\ a < y < b\}$, one has

$$(5.60)\qquad \begin{aligned}\int_{G\cap\Gamma_\varepsilon} |v_\sigma(t,X)|^2\, dX &= \int_{G\cap\Gamma_\varepsilon\cap R^2_{0,k}} |v_\sigma(t,X)|^2\, dX + \int_{\Gamma_\varepsilon\cap R^2_k} |v_\sigma(t,X)|^2\, dX\\ &\le C_\sigma^2 \int_{\Gamma_\varepsilon\cap R^2_k} e^{-2\mu r\sin\theta}\, r\, dr\, d\theta + o(1)\end{aligned}$$

for every fixed $k > r' > r$ by (5.54) and the local decay property for v_σ. Passage to the limit $t \to +\infty$ in (5.60) gives by (5.59),

$$(5.61)\qquad \int_{-\infty}^\infty \int_\varepsilon^{\pi-\varepsilon} |H_\sigma(\tau,\theta)|\, d\theta\, d\tau \le C_\sigma^2 \int_{\Gamma_\varepsilon\cap R^2_k} e^{-2\mu r\sin\theta}\, r\, dr\, d\theta$$

for every $k > r'$. Note that C_σ, μ and the left-hand side of (5.61) are independent of k. Now, $\sin\theta \ge \sin\varepsilon > 0$ for $\varepsilon \le \theta \le \pi - \varepsilon$ and hence

$$\int_{\Gamma_\varepsilon \cap R_k^2} e^{-2\mu r \sin\theta}\, r\, dr\, d\theta \leq \int_{\Gamma_\varepsilon \cap R_k^2} e^{-2\mu r \sin\varepsilon}\, r\, dr\, d\theta \tag{5.62}$$

$$\leq \int_k^\infty \int_\varepsilon^{\pi-\varepsilon} e^{-2\mu r \sin\varepsilon}\, r\, d\theta\, dr = (\pi - 2\varepsilon) \int_k^\infty e^{-2\mu r \sin\varepsilon}\, r\, dr\ .$$

But the last integral tends to zero when $k \to \infty$ with ε fixed. Thus (5.61) implies that $H_\sigma(\tau,\theta) \equiv 0$ in $R \times [\varepsilon,\pi-\varepsilon]$ and (5.56) follows since $\varepsilon \in [0,\pi/2]$ is arbitrary.

<u>Corollary 5.3</u>. For all $h \in L_2(G)$ such that supp $\hat{h}_\pm \subset \overline{\mathcal{O}}_{m,n}$ = closure of $\mathcal{O}_{m,n}$ one has the two relations

$$\hat{h}_\mp(p,q) = 2\pi \sum_{\ell=-n}^{m} c_\ell^\pm(X_{-\ell}(p,q))\ \hat{h}_\pm(X_{-\ell}(p,q))\ \frac{q}{q_{-\ell}} \tag{5.63}$$

for almost every $(p,q) \in R_0^2$ where $(p_{-\ell},q_{-\ell}) = X_{-\ell}(p,q)$.

<u>Proof</u>. The case where supp $\hat{h}_+ \subset \overline{\mathcal{O}}_{m,n}$ is considered first. In this case it will suffice to prove (5.63) for functions $h \in L_2(G)$ such that $\hat{h}_+ \in C_0^\infty(R_0^2 - \varepsilon)$ and supp $\hat{h}_+ = K \subset \mathcal{O}_{m,n}$ since such functions are dense in the subspace of $L_2(G)$ defined by supp $\hat{h}_+ \subset \overline{\mathcal{O}}_{m,n}$. For such functions the relation (5.56) and the Fourier representations (5.8) for $H(\tau,\theta)$ and (5.47), (5.48) for $H_\ell(\tau,\theta)$ imply that

$$\hat{h}_-(\omega\cos\theta,\omega\sin\theta) \tag{5.64}$$

$$= 2\pi \sum_{\ell=-n}^{m} c_\ell^+(X_{-\ell}(\omega\cos\theta,\omega\sin\theta))\hat{h}_+(X_{-\ell}(\omega\cos\theta,\omega\sin\theta))\ \frac{\omega\sin\theta}{\{\omega^2-(\omega\cos\theta-\ell)^2\}^{1/2}}$$

for almost every $(\omega,\theta) \in R_0 \times [0,\pi]$. Making the substitutions

$p = \omega \cos \theta$, $q = \omega \sin \theta$ in (5.64) gives (5.63) in the case $\text{supp}\ \hat{h}_+ \subset \mathcal{O}_{m,n}$.

The second case of (5.63) can be derived by calculating the asymptotic wave functions for $v(t,\cdot) = e^{-itA^{1/2}} h$ when $t \to -\infty$, using the method given above. A simpler derivation may be based on the relations $\psi_-(X,p,q) = \overline{\psi_+(X,-p,q)}$ and $c_\ell^-(p,q) = \overline{c_{-\ell}^+(-p,q)}$ of [5, (2.25) and (2.29)]. Indeed, if $\text{supp}\ \hat{h}_- \subset \mathcal{O}_{m,n}$ and $g(X) = \overline{h(X)}$ then these relations imply that $\hat{g}_\pm(p,q) = \overline{\hat{h}_\mp(p,q)}$ and hence relation (5.63) with the upper sign for g implies (5.63) with the lower sign for h.

<u>The Structure of $\hat{S}$</u>. It will be convenient to use the notation

$$g_{m,n}(P) = \chi_{m,n}(P)\, g(P) \tag{5.65}$$

where $\chi_{m,n}$ is the characteristic function of the set $\mathcal{O}_{m,n}$. Clearly, the operator $P_{m,n}$ in $L_2(R_0^2)$ defined by

$$P_{m,n}\, g = g_{m,n}\ , \quad m,n = 0,\ 1,\ 2,\cdots\ , \tag{5.66}$$

is an orthogonal projection and different operators of the family have orthogonal ranges. Moreover, the relation (5.15) implies that the family is complete because ε is a null set; i.e.,

$$\sum_{m,n=0}^{\infty} P_{m,n} = 1\ . \tag{5.67}$$

It follows that for all $g \in L_2(R_0^2)$ one has

$$\hat{S}\, g = \sum_{m,n=0}^{\infty} \hat{S}(g_{m,n})\ . \tag{5.68}$$

Thus $\hat{S}$ is completely determined by

Theorem 5.4. For all $g \in L_2(R_0^2)$ one has

$$(5.69) \qquad \hat{S}(g_{m,n}) = \sum_{\ell=-n}^{m} (\hat{S}\, g_{m,n})_{m-\ell,n+\ell}\ , \text{ and}$$

$$(5.70) \qquad (\hat{S}_{m,n})_{m-\ell,n+\ell}(p,q) = 2\pi \frac{q}{q_{-\ell}}\, c_\ell^+(X_{-\ell}(p,q))\, g_{m,n}(X_{-\ell}(p,q))\ .$$

Similarly, one has

$$(5.71) \qquad \hat{S}^*(g_{m,n}) = \sum_{\ell=-n}^{m} (\hat{S}^*\, g_{m,n})_{m-\ell,n+\ell}\ , \text{ and}$$

$$(5.72) \qquad (\hat{S}^*\, g_{m,n})_{m-\ell,n+\ell}(p,q) = 2\pi \frac{q}{q_{-\ell}}\, c_\ell^-(X_{-\ell}(p,q))\, g_{m,n}(X_{-\ell}(p,q))\ .$$

In particular, if supp $g \subset \overline{\mathcal{O}}_{m,n}$ then

$$(5.73) \qquad \operatorname{supp} \hat{S}\, g \cup \operatorname{supp} \hat{S}^* \subset \bigcup_{\ell=-n}^{m} \overline{\mathcal{O}}_{m-\ell,n+\ell}\ .$$

Proof. Equations (5.69), (5.70), (5.71) and (5.72) follow immediately from Corollary 5.3, the relations $\hat{h}_- = \hat{S}\,\hat{h}_+$, $\hat{h}_+ = \hat{S}^*\,\hat{h}_-$ and the observation that when supp $\hat{h}_+ \subset \overline{\mathcal{O}}_{m,n}$ then the ℓ^{th} term in the sum in (5.63) has its support in $\overline{\mathcal{O}}_{m-\ell,n+\ell}$. (5.73) follows from (5.69) and (5.71).

The unitarity of $\hat{S}$ and (5.70), (5.72) impose restrictions on the scattering coefficients $c_\ell^\pm$. To calculate them it will be convenient to calculate $\hat{S}^*$ and $\hat{S}$ directly from (5.70) and (5.72), respectively. This gives the following alternative representations of $\hat{S}^*$ and $\hat{S}$.

<u>Theorem 5.5</u>. For all $g \in L_2(R_0^2)$ one has

$$(\hat{S}^* g_{m,n})_{m-\ell,n+\ell}(p,q) = 2\pi\, \overline{c^+_{-\ell}(p,q)}\, g_{m,n}(X_{-\ell}(p,q)) \tag{5.74}$$

and similarly

$$(\hat{S}\, g_{m,n})_{m-\ell,n+\ell}(p,q) = 2\pi\, \overline{c^-_{-\ell}(p,q)}\, g_{m,n}(X_{-\ell}(p,q)) \ . \tag{5.75}$$

<u>Proof</u>. For all $f,g \in L_2(R_0^2)$ one has

$$\begin{aligned}(f,(\hat{S}^* g)_{m,n}) &= (f_{m,n},\hat{S}^* g) = (\hat{S}(f_{m,n}),g) \\ &= \sum_{\ell=-n}^{m} (\hat{S}(f_{m,n}),g_{m-\ell,n+\ell}) \\ &= 2\pi \sum_{\ell=-n}^{m} \int_{\mathcal{O}_{m-\ell,n+\ell}} \overline{c^+_{\ell}(X_{-\ell})\ f_{m,n}(X_{-\ell})}\ g_{m-\ell,n+\ell}\, \frac{q}{q_{-\ell}}\, dp\, dq\end{aligned} \tag{5.76}$$

by (5.70). On making the change of variables $(p',q') = (p_{-\ell},q_{-\ell}) = X_{-\ell}(p,q)$ in the last integral and noting that $q/q_{-\ell} = \partial(p',q')/\partial(p,q)$, one has

$$\begin{aligned}(f,(\hat{S}^*g)_{m,n}) &= 2\pi \sum_{\ell=-n}^{m} \int_{\mathcal{O}_{m,n}} \overline{c^+_{\ell}(p',q')f_{m,n}(p',q')}\ g_{m-\ell,n+\ell}(X_\ell(p',q'))dp'dq' \\ &= \int_{R_0^2} \overline{f(p,q)} \left[2\pi \sum_{\ell=-n}^{m} \overline{c^+_{\ell}(p,q)}\ g_{m-\ell,n+\ell}\ (X_\ell(p,q))\right] dp\ dq\end{aligned} \tag{5.77}$$

because $\operatorname{supp} g_{m-\ell,n+\ell}(X_\ell) \subset \overline{\mathcal{O}}_{m,n}$. Since $f \in L_2(R_0^2)$ is arbitrary, (5.77) implies that

$$(\hat{S}^* g)_{m,n}(p,q) = 2\pi \sum_{\ell=-n}^{m} \overline{c^+_{\ell}(p,q)}\ g_{m-\ell,n+\ell}(X_\ell(p,q)) \ . \tag{5.78}$$

To derive (5.74) note that for all ℓ, m, n, $\overline{m}$, $\overline{n} \geq 0$

$$(g_{\overline{m},\overline{n}})_{m-\ell,n+\ell} = \delta_{m-\ell,\overline{m}}\ \delta_{n+\ell,\overline{n}}\ g_{\overline{m},\overline{n}} \tag{5.79}$$

where δ_{jk} is the Kronecker symbol. Noting that $\delta_{m-\ell,\overline{m}}\ \delta_{n+\ell,\overline{n}}$ $= \delta_{\ell,m-\overline{m}}\ \delta_{\ell,\overline{n}-n} = \delta_{m-\overline{m},\overline{n}-n}\ \delta_{\ell,m-\overline{m}}$, (5.78) and (5.79) imply

$$\begin{aligned}(\hat{S}^*\ g_{\overline{m},\overline{n}})_{m,n}(p,q) &= 2\pi\ \delta_{m-\overline{m},\overline{n}-n} \sum_{\ell=-n}^{m} \overline{c_\ell^+(p,q)}\ \delta_{\ell,m-\overline{m}}\ g_{\overline{m},\overline{n}}(X_\ell(p,q)) \\ &= 2\pi\ \delta_{m-\overline{m},\overline{n}-n}\ \overline{c^+_{m-\overline{m}}(p,q)}\ g_{\overline{m},\overline{n}}(X_{m-\overline{m}}(p,q))\ .\end{aligned} \tag{5.80}$$

This clearly is zero unless $\overline{m} - m = n - \overline{n} = \ell$ where $-n \leq \ell \leq m$, which implies (5.71). Moreover, setting $(m,n) = (\overline{m} - \ell, \overline{n} + \ell)$ in (5.80) gives (5.74). The proof of (5.75) is obtained by the same method, beginning with (5.72).

The two representations of $\hat{S}$ and $\hat{S}^*$ of Theorems 5.4 and 5.5 hold for arbitrary $g \in L_2(R_0^2)$. It follows that the scattering coefficients must satisfy the relations

$$q\ c_\ell^\pm(X_{-\ell}(p,q)) = q_{-\ell}\ \overline{c_{-\ell}^\mp(p,q)} \tag{5.81}$$

for all $(p,q) \in \mathcal{O}_{m-\ell,n+\ell}$. Moreover, the unitarity of $\hat{S}$ and Theorems 5.4 and 5.5 imply

<u>Theorem 5.6</u>. The scattering coefficients $c_\ell^\pm$ of the R-B waves $\psi_\pm(X,p,q)$ satisfy the identities

$$\sum_{\ell=-n}^{m} \overline{c_\ell^\pm(p,q)}\ c_{\ell-k}^\pm\ (X_k(p,q))q_\ell = (2\pi)^{-2}\ q\ \delta_{k,0}\ , \text{ and} \tag{5.82}$$

$$(5.83)\qquad \sum_{\ell=-n}^{m} \overline{c^{\pm}_{-\ell}(X_\ell(p,q))}\; c^{\pm}_{k-\ell}(X_\ell(p,q))q_\ell^{-1} = (2\pi)^{-2}q^{-1}\;\delta_{k,0}$$

for all $(p,q) \in \mathcal{O}_{m,n}$ and all k such that $-n \le k \le m$.

These properties may be verified by simple calculations using the relations

$$(5.84)\qquad (\hat{S}(f_{m,n}),\hat{S}(g_{m-k,n+k})) = \delta_{k,0}(f_{m,n},g_{m,n})\ , \text{ and}$$

$$(5.85)\qquad (\hat{S}^*(f_{m,n}),\hat{S}^*(g_{m-k,n+k})) = \delta_{k,0}(f_{m,n},g_{m,n})$$

and the constructions of $\hat{S}$ and $\hat{S}^*$ described in Theorems 5.4 and 5.5. Relation (5.83) also follows from relations (5.81) and (5.82).

It is well known in the theories of scattering by potentials and by bounded obstacles that the S-matrix $\hat{S}$ is a direct integral of a family of unitary operators $\hat{S}(\omega)$ that act on the "energy shell" $p^2 + q^2 = \omega^2$. The analogous property of the S-matrices for diffraction gratings is evident from Theorem 5.4 and the properties of the mappings X_ℓ. The operator $\hat{S}(\omega)$ in this case is given by (cf. (5.75))

$$(5.86)\qquad \hat{S}(\omega)\; g(\omega\cos\theta,\omega\sin\theta) = 2\pi \sum_{\ell=-m}^{n} \overline{c^{-}_{\ell}(\omega\cos\theta,\omega\sin\theta)}\; g(X_\ell(\omega\cos\theta,\omega\sin\theta))$$

when $\operatorname{supp} g \subset \overline{\mathcal{O}}_{m,n}$. If $s(\theta) = g(\omega\cos\theta,\omega\sin\theta)$ is an arbitrary function with $\operatorname{supp} s \subset \{\theta : (\omega\cos\theta,\omega\sin\theta) \in \overline{\mathcal{O}}_{m,n}\}$ then (5.86) can be written

$$(5.87)\qquad (\hat{S}(\omega)s)(\theta) = 2\pi \sum_{\ell=-m}^{n} \overline{c^{-}_{\ell}(\omega\cos\theta,\omega\sin\theta)}\; s(\theta_\ell)$$

where $\theta_\ell = \theta_\ell(\omega,\theta)$ is defined as the unique angle such that $0 \le \theta_\ell \le \pi$ and

$$X_\ell(\omega \cos \theta, \omega \sin \theta) = (\omega \cos \theta_\ell, \omega \sin \theta_\ell) \quad (5.88)$$

For general $s \in L_2(0,\pi)$, $\hat{S}(\omega)s$ is obtained from (5.87), (5.88) by superposition. The unitarity of $\hat{S}(\omega)$ in $L_2(0,\pi)$ can be verified by direct calculation using (5.87), the analogue for $\hat{S}^*(\omega)$ and Theorem 5.6.

§6. The Scattering of Signals by Diffraction Gratings.

The results of §4 and §5 are applicable to the echoes that are produced when signals generated by localized sources are scattered by a diffraction grating. The structure of such echoes is analyzed in this section. Most of the section deals with the case, often realized in applications, where the sources are far from the grating. In particular, it is shown that in this case the influence of the grating on the echoes is completely described by the S-matrix.

It will be assumed that the sources of the signals are localized near a point $(0,y_0) \in G$ and act during a time interval $T \leq t \leq 0$. The resulting wave field $u(t,X)$ is then characterized by its initial values $u(0,X)$, $D_t u(0,X)$ in G. To make explicit their dependence on y_0 the initial values will be assumed to have the form

$$(6.1) \qquad \begin{aligned} u(0,X) &= f(X,y_0) \equiv f_0(x,y-y_0) , \\ D_t u(0,X) &= g(X,y_0) \equiv g_0(x,y-y_0) \end{aligned}$$

for all $X = (x,y) \in G$ where $y_0 \geq 0$,

$$(6.2) \qquad f_0 \in L_2^{1,\text{com}}(G) \ , \ g_0 \in L_2^{\text{com}}(G) \ ,$$

and $f(X,y_0) \equiv g(X,y_0) \equiv 0$ for $(x,y-y_0) \notin G$. Note that for $y_0 \geq 0$ one has $f(\cdot,y_0) \in D(A^{1/2})$, $g(\cdot,y_0) \in L_2(G)$ and hence $u(t,X)$ is a solution wFE in G. The functions $f(\cdot,y_0)$, $g(\cdot,y_0)$ will also be used as initial values for free waves in R^2 and for wave fields in the degenerate grating domain R_0^2. In each case the domain under consideration will be clear from the context or will be stated explicitly. For brevity, the

coordinate y_0 will be suppressed except in places where the y_0-dependence is under discussion.

<u>The Signal Wave Field</u>. In the absence of a diffraction grating the initial state f_0, g_0 will generate a signal wave field $u_s(t,X)$ in R^2. The first derivatives of $u_s(t,X)$ have asymptotic wave functions [3, Theorem 2.10]

$$(6.3) \qquad D_k\, u_s(t,X) = r^{-1/2}\, s_k(r-t,\theta) + o(1),\ k = 0,1,2 ,$$

where the waveforms $s_k(\tau,\theta) \in L_2(R \times [-\pi,\pi])$ are given by

$$(6.4) \qquad s_0(\tau,\theta) = \mathrm{Re}\left\{\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega}\, \hat{h}_0(\omega \cos\theta, \omega \sin\theta)(-i\omega)^{1/2}\, d\omega\right\} ,$$

$s_1(\tau,\theta) = -s_0(\tau,\theta) \cos\theta$, $s_2(\tau,\theta) = -s_0(\tau,\theta) \sin\theta$ and the terms $o(1) \to 0$ in $L_2(R^2)$ when $t \to \infty$. The function

$$(6.5) \qquad \hat{h}_0(P) = \Phi\, g_0(P) - i\omega(P)\, \Phi\, f_0(P)$$

where Φ denotes the Fourier transform in $L_2(R^2)$. In particular, the Fourier transform in $L_2(R \times [-\pi,\pi])$ of the signal waveform $s_0(\tau,\theta)$ is

$$(6.6) \qquad \hat{s}_0(\omega,\theta) = \frac{1}{2}\, (-i\omega)^{1/2}\, \hat{h}_0(\omega \cos\theta, \omega \sin\theta) .$$

It can be verified that if f_0 and g_0 are real-valued then $\hat{s}_0(-\omega,\theta) = \overline{\hat{s}_0(\omega,\theta)}$ and hence (6.6) generates a real-valued signal.

When y_0 is large the signal arriving at the grating surface is described by the signal waveform $s_0(\tau,\theta)$ through (6.3). The problem of signal design is to construct a source or "transmitter" whose waveform $s_0(\tau,\theta)$ approximates a prescribed function. The solution of this problem is the task of the transmitter design engineer.

<u>The Echo Wave Fields</u>. In the presence of a diffraction grating with domain G the initial state f,g will generate a total wave field $u(t,X)$ whose asymptotic behavior for $t \to +\infty$ is described by Theorem 4.3. In particular,

$$D_0\, u(t,X) = r^{-1/2} F_0(r-t,\theta) + o(1) \text{ in } L_2(G),\ t \to +\infty \,, \tag{6.7}$$

where $F_0(\tau,\theta) \in L_2(R \times [0,\pi])$ is defined by

$$F_0(\tau,\theta) = \operatorname{Re}\left\{\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega}\, \hat{h}_-(\omega\cos\theta, \omega\sin\theta)(-i\omega)^{1/2}\, d\omega\right\} \tag{6.8}$$

and

$$\hat{h}_-(P) = \hat{g}_-(P) - i\omega(P)\, \hat{f}_-(P)\ . \tag{6.9}$$

The echo wave field $u_e(t,X)$ is defined by

$$u_e(t,X) = u(t,X) - u_s(t,X)\ ,\ t \geq 0\ ,\ X \in G\ . \tag{6.10}$$

Thus the echo is described for large t by

$$D_0 u_e(t,X) = r^{-1/2} e_0(r-t,\theta) + o(1) \text{ in } L_2(G),\ t \to +\infty \,, \tag{6.11}$$

where $e_0 = F_0 - s_0 \in L_2(R \times [0,\pi])$ is given by

$$e_0(\tau,\theta) = \operatorname{Re}\left\{\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega}\, \hat{h}_-^{sc}(\omega\cos\theta, \omega\sin\theta)(-i\omega)^{1/2}\, d\omega\right\} \tag{6.12}$$

with

$$\hat{h}_-^{sc}(P) = \hat{h}_-(P) - \hat{h}(P) = \hat{g}_-^{sc}(P) - i\omega(P)\, \hat{f}_-^{sc}(P)\ . \tag{6.13}$$

The last functions can be written in terms of the R-B diffracted plane waves

$$(6.14) \qquad \psi_-(X,P) = \psi^{inc}(X,P) + \psi_-^{sc}(X,P)$$

as [5, (2.24)]

$$(6.15) \qquad \hat{g}_-^{sc}(P) = \int_G \overline{\psi_-^{sc}(X,P)}\, g(X)\, dX$$

with the analogous representation for $\hat{f}_-^{sc}$.

<u>The Echoes of Signals from Remote Sources</u>. Equations (6.12) - (6.15) provide a construction of the echo due to an arbitrary distribution of sources. The principal goal of this section is to determine how this construction may be simplified when the sources are far from the grating; i.e., $y_0 \to \infty$. To this end recall the decomposition of Lemma 5.1. Substituting equation (5.18) in (6.15) gives

$$(6.16) \quad \hat{g}_-^{sc}(p,q) = 2\pi \sum_{\ell=-n}^{m} \overline{c_\ell^-(p,q)}\, \hat{g}(p_\ell,-q_\ell) + \rho_{m,n}(p,q) \ , \ (p,q) \in \mathcal{O}_{m,n} \ ,$$

where $\hat{g} = \Phi\, g$ is the Fourier transform in $L_2(R^2)$ and

$$(6.17) \quad \rho(p,q) \equiv \rho(p,q;g) = \int_G \overline{\sigma_-(X,p,q)}\, g(X)\, dX \ , \ (p,q) \in R_0^2 - E \ .$$

Note that if the unitary operator $R : L_2(R^2) \to L_2(R^2)$ is defined by

$$(6.18) \qquad R\, f(x,y) = f(x,-y)$$

then

$$(6.19) \qquad (\Phi\, R\, f)(p,q) = (R\, \hat{f})(p,q) = \hat{f}(p,-q) \ .$$

Hence (6.16) implies that for all $(p,q) \in \mathcal{O}_{m,n}$ one has

$$(6.20) \qquad \begin{aligned} \hat{g}_-^{sc}(P) &= 2\pi \sum_{\ell=-n}^{m} \overline{c_\ell^-(P)}\ (R\, \hat{g})(P_\ell) + \rho_{m,n}(P) \\ &= 2\pi \sum_{\ell=-m}^{n} \overline{c_{-\ell}^-(P)}\ (R\, \hat{g})_{m+\ell,n-\ell}(X_{-\ell}(P)) + \rho_{m,n}(P) \\ &= \sum_{\ell=-m}^{n} (\hat{S}(R\, \hat{g})_{m+\ell,n-\ell})_{m,n}(P) + \rho_{m,n}(P) \\ &= (\hat{S}\, R\, \hat{g})_{m,n}(P) + \rho_{m,n}(P) \end{aligned}$$

by Theorems 5.4 and 5.5. Proceeding in the same way with $\omega(P)\ \hat{f}_-^{sc}(P)$ and recalling that $\omega(P) = \omega(P_\ell)$ one finds

$$(6.21) \qquad \hat{h}_-^{sc}(P) = (\hat{S}\, R\, \hat{h})(P) + \rho(P;h) \ , \ P \in R_0^2 - E \ ,$$

where

$$(6.22) \qquad \rho(P;h) = \rho(P;g) - i\,\omega(P)\ \rho(P;f) \ .$$

The estimate (5.19) of Lemma 5.1 clearly implies that $\rho(P;h(\cdot,y_0)) \to 0$ when $y_0 \to \infty$, uniformly for P in any compact subset of $R_0^2 - E$. This result is not strong enough to yield a corresponding estimate of the echo waveform $e_0(\tau,\theta)$ defined by (6.12) and it is natural to conjecture that $\rho(\cdot;h(\cdot,y_0)) \to 0$ in $L_2(R_0^2)$ when $y_0 \to \infty$. Unfortunately, if one assumes only that A(G) admits no surface waves then this property does not follow from the results of [5,6] because no information was obtained concerning the behavior of $\psi_-(X,P)$ for P near

the exceptional set E. However, in those cases where the analytic continuation of the resolvent of A_p has no singularities on $\sigma(A_p)$ (i.e., $\Sigma_p \cap \sigma(A_p) = \phi$ for every $p \in (-1/2,1/2]$) the limiting absorption theorem [5, Corollary 4.17] is valid on all of $\sigma(A_p)$ (see [5, Theorem 4.15]) and [5, Theorem 6.1] can be improved to state that $\psi_\pm(\cdot,P)$ exists and $P \to \psi_\pm(\cdot,P) \in L_2^{1,\ell oc}(\Delta,G)$ is continuous for all $P \in \overline{R_0^2}$. This improvement of [5, Theorem 6.1] implies

Theorem 6.1. Let $A(G)$ have no surface waves and, in addition, assume that

$$(6.23) \qquad \Sigma_p \cap \sigma(A_p) = \phi \text{ for all } p \in (-1/2,1/2] \ .$$

Then for every $g_0 \in L_2(G)$ one has

$$(6.24) \qquad \hat{g}_-^{sc}(\cdot,y_0) = \hat{S}\, R\, \hat{g}(\cdot,y_0) + o(1) \text{ in } L_2(R_0^2),\ y_0 \to \infty \ .$$

Similarly, for all $f_0 \in L_2^1(G)$ one has

$$(6.25) \qquad \omega(\cdot)\, \hat{f}_-^{sc}(\cdot,y_0) = \omega(\cdot)\, \hat{S}\, R\, \hat{f}(\cdot,y_0) + o(1) \text{ in } L_2(R_0^2),\ y_0 \to \infty \ .$$

The proof of Theorem 6.1 will be based on the following extension of Lemma 5.1.

Lemma 6.2. Under the hypotheses of Theorem 6.1, for every compact set $K \subset \overline{R_0^2}$ and every $r' > r > h$ there is a constant $C = C(K,r,r')$ such that

$$(6.26) \qquad |\sigma_\pm(X,P)| \le C \text{ for all } X \in R_r^2,\text{ and } P \in K \ .$$

Proof of Lemma 6.2. It clearly suffices to prove the lemma for the case $K = \overline{\mathcal{O}}_{m,n}$. On examining the proof of Lemma 5.1 one finds that the continuity of $P \to \psi_\pm(\cdot,P)$ for all $P \in \overline{R_0^2}$ implies that (5.24) holds

for $K = \overline{\mathcal{O}}_{m,n}$. Moreover, (5.25) holds for all $P \in \overline{\mathcal{O}}_{m,n}$ with $\mu_+ = \mu_- = 0$. Thus (6.26) follows from (5.27) with $\mu(K) = 0$.

Proof of Theorem 6.1. Note first that if the translation operator $T_{y_0} : L_2(G) \to L_2(G)$ is defined for each $y_0 \geq 0$ by $T_{y_0} g_0 = g(\cdot, y_0)$ then (6.24) is equivalent to the statement that

$$\text{s-lim}_{y_0 \to \infty} (\Phi_- - \Phi - \hat{S} R \Phi) T_{y_0} = 0 . \tag{6.27}$$

Moreover, the family of operators in (6.27) is uniformly bounded for all $y_0 \geq 0$. Hence by a familiar density argument (cf. [3, proof of Theorem 2.6]) it will suffice to establish (6.24) for all g_0 in a dense subset of $L_2(G)$. The set $C_0^\infty(G)$ will be chosen for this purpose. Thus the proof will be completed by showing that if $g_0 \in C_0^\infty(G)$ and

$$\begin{aligned} \rho(P, g(\cdot, y_0)) &= \hat{g}_-^{sc}(P, y_0) - \hat{S} R \hat{g}(\cdot, y_0)(P) \\ &= \int_G \overline{\sigma_-(X,P)}\, g(X, y_0)\, dX , \quad P \in R_0^2 , \end{aligned} \tag{6.28}$$

then

$$\lim_{y_0 \to \infty} \int_{R_0^2} |\rho(P, g(\cdot, y_0))|^2 \, dP = 0 . \tag{6.29}$$

To prove (6.29) it will be convenient to decompose R_0^2 as the disjoint union

$$R_0^2 = D(\gamma) \cup (D'(\gamma) \cap E_\delta) \cup (D'(\gamma) - E_\delta) \tag{6.30}$$

where

$$D(\gamma) = R_0^2 \cap \{P : |P| \geq \gamma\} ,$$

(6.31)

$$D'(\gamma) = R_0^2 \cap \{P : |P| \leq \gamma\} , \text{ and}$$

$$E_\delta = R_0^2 \cap \{P : \text{dist } (P, \ ) \leq \delta\} .$$

With this notation the integral in (6.29) can be written

$$(6.32) \qquad \int_{R_0^2} |\rho(P,g(\cdot,y_0))|^2 \, dP = I_1(\gamma,y_0) + I_2(\gamma,\delta,y_0) + I_3(\gamma,\delta,y_0)$$

where

$$(6.33) \qquad I_1(\gamma,y_0) = \int_{D(\gamma)} |\rho(P,g(\cdot,y_0)|^2 \, dP ,$$

$$(6.34) \qquad I_2(\gamma,\delta,y_0) = \int_{D'(\gamma)\cap E_\delta} |\rho(P,g(\cdot,y_0))|^2 \, dP , \text{ and}$$

$$(6.35) \qquad I_3(\gamma,\delta,y_0) = \int_{D'(\gamma)-E_\delta} |\rho(P,g(\cdot,y_0))|^2 \, dP .$$

To estimate $I_1(\gamma,y_0)$ note that (5.18) implies that $(\Delta + |P|^2)\, \sigma\_(X,P) = 0$ for all $X \in G$, $P \in R_0^2$. Thus integrating by parts in (6.28) gives

$$\rho(P,g(\cdot,y_0)) = -|P|^{-2} \int_G \overline{\Delta\sigma\_(X,P)} \, g(X,y_0) \, dX$$

(6.36)

$$= -|P|^{-2} \int_G \overline{\sigma\_(X,P)} \, \Delta g(X,y_0) \, dX$$

$$= -|P|^2 \{\Delta g(\cdot,y_0)\hat{\ }_-^{,sc}(P) - \hat{S} \, R(\Delta g(\cdot,y_0))\hat{\ }(p)\}$$

$$= -|P|^{-2} \{(\Delta g(\cdot,y_0))\hat{\ }_-(P) - (\Delta g(\cdot,y_0))\hat{\ }(P) - \hat{S} \, R(\Delta g(\cdot,y_0))\hat{\ }(P)\}$$

On squaring (6.36), integrating over $D(\gamma)$ and using the inequality $|z_1 + z_2 + z_3|^2 \le 4(|z_1|^2 + |z_2|^2 + |z_3|^2)$ one finds

$$(6.37)\quad \begin{aligned} I_1(\gamma,y_0) &\le \int_{D(\gamma)} |P|^{-4} \, |(\Delta g)\hat{}_- - (\Delta g)\hat{} - \hat{S}\, R(\Delta g)\hat{}\,|^2 \, dP \\ &\le 4\gamma^{-4}(\|\Phi_-(\Delta g(\cdot,y_0))\|^2 + \|\Phi(\Delta g(\cdot,y_0)\|^2 + \|\hat{S}\, R(\Delta g(\cdot,y_0))\|^2) \\ &\le 12\gamma^{-4} \, \|\Delta g(\cdot,y_0)\|^2 = 12\gamma^{-4} \, \|\Delta g_0\|^2_{L_2(G)} \end{aligned}$$

for all $y_0 \ge 0$. In particular, $I_1(\gamma,y_0)$ is small for large γ, uniformly in $y_0 \ge 0$.

Now consider $I_2(\gamma,\delta,y_0)$. Lemma 6.2 and equation (6.28) imply that for all $P \in D'(\gamma)$ one has

$$(6.38)\quad \begin{aligned} |\rho(P,g(\cdot,y_0)| &\le C(D'(\gamma),r,r') \int_G |g_0(x,y-y_0)| \, dx\, dy \\ &= C_1(\gamma,r,r') \int_G |g_0(X)| \, dX = C_2(g_0,\gamma,r,r') \ . \end{aligned}$$

Combining this and (6.34) gives

$$(6.39)\qquad I_2(\gamma,\delta,y_0) \le C_2^2(g_0,\gamma,r,r') \, |D'(\gamma) \cap E_\delta| \ ,$$

for all $y_0 \ge 0$, where $|M|$ denotes the Lebesgue measure of a set $M \subset R^2$. Finally, note that Lemma 5.1 implies that $\rho(P,g(\cdot,y_0)) \to 0$ when $y_0 \to \infty$, uniformly for $P \in D'(\gamma) - E_\delta$, when $\gamma > 0$ and $\delta > 0$ are fixed. Thus

$$(6.40)\qquad \lim_{y_0 \to \infty} I_3(\gamma,\delta,y_0) = 0 \ , \ \gamma \text{ and } \delta \text{ fixed} \ .$$

To complete the proof of (6.29) let $\varepsilon > 0$ be given and use (6.37) to choose a $\gamma = \gamma_0 = \gamma_0(\varepsilon, g_0) > 0$ such that $I_1(\gamma, y_0) < \varepsilon/3$. Next use (6.39) with $\gamma = \gamma_0(\varepsilon, g_0)$ fixed to choose $\delta = \delta_0 = \delta_0(\varepsilon, g_0) > 0$ so small that $I_2(\gamma_0, \delta_0, y_0) < \varepsilon/3$. Both of these estimates hold uniformly for all $y_0 \geq 0$. Finally, choose $Y_0 = Y_0(\varepsilon, g_0)$ so large that $I_3(\gamma_0, \delta_0, y_0) < \varepsilon/3$ for all $y_0 \geq Y_0$. This is possible by (6.40). With these choices (6.32) implies that

$$(6.41) \qquad \int_{R_0^2} |\rho(P, g(\cdot, y_0))|^2 \, dP < \varepsilon \text{ for all } y_0 \geq Y_0(\varepsilon, g_0) \; ,$$

which proves (6.29) and therefore (6.24). Finally, to prove (6.25) one notes that if $f(\cdot, y_0) \in L_2^1(G)$ then $\omega(P)\, \hat{f}_-(P, y_0) \in L_2(R_0^2)$ and the preceding argument can be applied to this function. This completes the proof of Theorem 6.1.

<u>An Estimate of the Echo Waveform</u>. Under the hypotheses of Theorem 6.1 one has the estimate

$$(6.42) \qquad \begin{aligned} \hat{h}_-^{sc}(\cdot, y_0) &= \hat{g}_-^{sc}(\cdot, y_0) - i\omega(\cdot)\, \hat{f}_-^{sc}(\cdot, y_0) \\ &= \hat{S}\, R\, \hat{g}(\cdot, y_0) - i\omega(\cdot)\, \hat{S}\, R\, \hat{f}(\cdot, y_0) + o(1) \\ &= \hat{S}\, R\, \hat{h}(\cdot, y_0) + o(1) \end{aligned}$$

where $o(1) \to 0$ in $L_2(R_0^2)$ when $y_0 \to \infty$. Moreover, the mapping $\hat{h}_-^{sc} \in L_2(R_0^2) \to e_0 \in L_2(R \times [0,\pi])$ defined by (6.12) is bounded with bound 1 [3, (2.84)]. It follows that

$$(6.43) \quad e_0(\tau, \theta) = \text{Re} \left\{ \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i\tau\omega} (\hat{S}\, R\, \hat{h})(\omega \cos\theta, \omega \sin\theta, y_0)(-i\omega)^{1/2} \, d\omega \right\} + o(1)$$

where $o(1) \to 0$ in $L_2(R \times [0,\pi])$ when $y_0 \to \infty$. Now

$$(6.44)\quad \begin{aligned} \hat{h}(p,q,y_0) &= \hat{g}(p,q,y_0) - i\omega(p,q)\,\hat{f}(p,q,y_0) \\ &= e^{iqy_0}[\hat{g}_0(p,q) - i\omega(p,q)\,\hat{f}_0(p,q)] \\ &= e^{iqy_0}\,\hat{h}_0(p,q) \end{aligned}$$

and hence by (6.6)

$$(6.45)\quad \begin{aligned} (-i\omega)^{1/2} R\,\hat{h}(\omega\cos\theta,\omega\sin\theta,y_0) &= (-i\omega)^{1/2}\,\hat{h}(\omega\cos\theta,-\omega\sin\theta,y_0) \\ &= e^{-i\omega y_0 \sin\theta}(-i\omega)^{1/2}\,\hat{h}_0(\omega\cos\theta,-\omega\sin\theta) \\ &= 2\,e^{-i\omega y_0\sin\theta}\,\hat{s}_0(\omega,-\theta)\ . \end{aligned}$$

Combining (6.43) and (6.45) gives

$$(6.46)\quad e_0(\tau,\theta) = \mathrm{Re}\left\{\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty e^{i\tau\omega}\,e^{-i\omega y_0\sin\theta}\,\hat{S}(\omega)\,\hat{s}_0(\omega,-\theta)\,d\omega\right\} + o(1)\ .$$

Thus under the hypotheses of Theorem 6.1 the echo waveform is determined by the signal waveform, the S-matrix for the grating and the range parameter y_0, with an error that tends to zero in energy when $y_0 \to \infty$.

<u>Pulsed Beam Signals</u>. For many applications it is desirable to have a transmitter whose waveform $s_0(\tau,\theta)$ is sharply limited in both direction and frequency. The relation (6.6) shows that this could be achieved by choosing f_0 and g_0 such that supp $\hat{h}_0 = RK$ where $K \subset \mathcal{O}_{m,n}$ and m and n are suitably chosen. Of course, this condition cannot be satisfied with sources that are confined to a compact set, since $h_0(P)$ is then analytic. However, it may be possible to choose f_0 and g_0 such that

$$\hat{h}_0(p,q) = a(p,q) + b(p,q) \tag{6.47}$$

where

$$\operatorname{supp} a = RK \subset R\mathcal{O}_{m,n} , \tag{6.48}$$

$$\hat{s}_0^a(\omega,\theta) = \frac{1}{2}(-i\omega)^{1/2} a(\omega\cos\theta, \omega\sin\theta) \tag{6.49}$$

defines the desired waveform s_0^a and

$$\|b\|_{L_2(R^2)} < \varepsilon . \tag{6.50}$$

If this transmitter design problem has been solved then the corresponding echoes will satisfy

$$e_0(\tau,\theta) = e_0^a(\tau,\theta) + o_1 + o_2 \tag{6.51}$$

where

$$e_0^a(\tau,\theta) = \operatorname{Re}\left\{\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty e^{i\tau\omega} e^{-i\omega y_0 \sin\theta} \hat{S}(\omega) \hat{s}_0^a(\omega,-\theta) d\omega\right\} \tag{6.52}$$

while

$$\|o_1\|_{L_2(R\times[0,\pi])} < \varepsilon \text{ for all } y_0 \geq 0 \text{ , and} \tag{6.53}$$

$$\lim_{y_0\to\infty} \|o_2\|_{L_2(R\times[0,\pi])} = 0 . \tag{6.54}$$

<u>Angular Dispersion of Echoes from Gratings</u>. The notation

$$\Gamma_\ell = \{P = (\omega\cos\theta, \omega\sin\theta) : \omega > 0 \text{ and } \alpha_\ell \leq \theta \leq \beta_\ell\} \tag{6.55}$$

will be used to denote the smallest sector such that $K_\ell \subset \Gamma_\ell$, $-n \leq \ell \leq m$.

The hypothesis $K = K_0 \subset \mathcal{O}_{m,n}$ implies that the sectors Γ_ℓ are disjoint and

(6.56) $$\bigcup_{\ell=-n}^{m} \Gamma_\ell \subset R_0^2 .$$

Moreover, (6.48), (6.49) and (6.52) and Theorem 5.4 imply that one has

(6.57) $$\operatorname{supp} r^{-1/2} e_0^a(r-t,\theta) \subset \bigcup_{\ell=-n}^{m} \Gamma_\ell$$

for all $t > 0$. Thus, apart from the error terms in (6.51), the echo waveform is concentrated in the sectors Γ_ℓ. Note that in the case of a degenerate grating with Neumann (resp., Dirichlet) boundary condition one has $\hat{S} = 1$ (resp., $\hat{S} = -1$) and hence $r^{-1/2} e_0^a(r-t,\theta) = \pm r^{-1/2} s_0^a(r-t,\theta)$ has support in Γ_0. This is a well-known property of the specular reflection of a beam by a plane. In the case of a non-degenerate grating, where $\hat{S} \neq \pm 1$, one has only (6.57) and secondary reflected beams will appear in the sectors Γ_ℓ, $\ell \neq 0$. Their waveforms can be calculated explicitly using (6.52) and (5.87). They are distortions of the signal waveform $s_0(\tau,\theta)$ whose forms are determined by the scattering coefficients $c_\ell^-(\omega\cos\theta,\omega\sin\theta)$. This phenomenon of the angular dispersion of pulsed beams by diffraction gratings is the counterpart for transient wavefields of the phenomenon of the diffraction of monochromatic beams into the higher order grating directions.

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Abstract.

Scattering theory is developed for plane diffraction gratings. The author's theory of Rayleigh-Bloch wave expansions is used to construct wave operators and a scattering operator for such gratings. For gratings that admit no surface waves, transient wave fields near the gratings are shown to behave for large times like free waves and corresponding asymptotic wave functions are calculated. These results are applied to analyze the echoes from gratings of signals due to localized sources. Finally, the echoes of sources remote from the grating are estimated and shown to be completely characterized by the S-matrix and the signal waveform.

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